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BEST APPROXIMATION PAIR OF TWO SKEW LINES VIA AN ANDERSON-DUFFIN FORMULA

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Abstract

This is a paper on geometry in the usual real Euclidean space. We treat the distance between two skew lines, by using a very beautiful instance of the parallel sum of two matrices. We display the two points where the shortest distance between two skew lines is achieved. This paper offers the reader a great amount of facts aiming to rise awareness about delicate questions - when dealing with infinite dimensional spaces: on closed sum of subspaces; on the non-linearity of the projector operator; and on the necessity of the sum of (closed) subspaces to be closed in order to guarantee the existence of the Moore-Penrose generalized inverse of the sum of two projectors. In making use of approximation theory results, this text is sprinkled with observations regarding concepts stemming from linear algebra to functional analysis to topology.

1. Introduction

In this paper, we are going to study the point which is the projection of a point onto a line and the unique pair of best approximation points of two skew lines, in the real Euclidean space $\mathbb{R}^3$.

The objective of this paper is twofold:

- to apply an interesting formula of linear algebra to a problem of space geometry;

- to pave the way for dealing with more complex (and complicated) problems of approximation theory in higher dimensional real spaces.

We use the Euclidean distance, this fact allowing the use of translations and their reverses - an instance of conjugacy [17].

We present the distance point/line and the distance line/line, by using tools of linear algebra. However, we go a bit further and, in an Appendix, we deal with pertinent questions that are transversal to linear algebra, functional analysis and topology.

In this paper, appropriate use is made of a well known generalized inverse of a matrix: the Moore-Penrose inverse.

Given a real matrix *A*, the Moore-Penrose inverse $A^{\dagger}$ of *A* is the unique matrix that satisfies [6, p. 9] the following four relations:

$$AA^{\dagger}A = A, \quad A^{\dagger}AA^{\dagger} = A^{\dagger}, \quad (AA^{\dagger})^{T} = AA^{\dagger}, \quad (A^{\dagger}A)^{T} = A^{\dagger}A,$$

where *T* stands for the transposition of matrices.

For computing the Moore-Penrose inverse, there are formulas and algorithms as was extensively detailed in [8]. Here we mention the Decell algorithm [9] and a procedure by Marmolejo and Villegas [16, p. 109, Theorem 5.20] and [15].

Now we are going to present a jewel of linear algebra: the Anderson-Duffin formula [18, 1].

Let **M** and **N** be subspaces of the same vector space. Then the orthogonal projection onto $\mathbf{M} \cap \mathbf{N}$ is given by [18, p. 441]

$$P_{\mathbf{M}\cap\mathbf{N}} = 2P_{\mathbf{M}}(P_{\mathbf{M}} + P_{\mathbf{N}})^{\dagger}P_{\mathbf{N}},$$

with [18, p. 430]

$$P_{\mathbf{M}} = M(M^{T}M)^{-1}M^{T}$$

and

$$P_{\mathbf{N}} = N(N^{T}N)^{-1}N^{T},$$

where the columns of M are a basis for the subspace **M**, and the columns of N are a basis for the subspace **N**, and $P_{\mathbf{M}}$ and $P_{\mathbf{N}}$ are orthogonal projections onto the subspaces **M** and **N**, respectively.

The formula $P_{\mathbf{M}} = M(M^T M)^{-1} M^T$ is independent of the choice of M **-** just as long as its columns constitute some basis for **M** [18, p. 430].

As we deal with lines, which in this paper are subspaces of $\mathbb{R}^3$, we deal with planes whose intersection is the line under consideration.

There are situations where some abuse of notation occurs. The sign $:=$ is used to identify these situations. We consider the orthonormal referential $\{O, (\vec{e}_1, \vec{e}_2, \vec{e}_3)\}$ and, permitted by convenient isomorphism, we express points and vectors in several ways, out of necessity and convenience at each moment.

The inner product $\vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ of two vectors is used, and the Euclidean norm of $\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$ is $\| \vec{u} \| = \sqrt{\vec{u} \bullet \vec{u}}$.

This paper is organized as follows: In Section 2, we deal with the distance from a point to a line. In Section 3, we treat the distance between two skew lines. In Section 4, an Appendix contains a large number of observations concerning some delicate questions when dealing with infinite dimensional spaces. Finally, in Section 5, some conclusions and remarks are offered.

2. Distance from a Point to a Line

We are given a line and an external point. We construct the projection of the point onto the line, one (out of two) vector connecting the given point and the obtained projection and, finally, we show the distance between the point and the line.

In order to use the Anderson-Duffin formula, where subspaces are involved, we need to displace the given line to the origin of the coordinates. So, first we consider that the given line contains the origin of the coordinates, and then we study the general case.

2.1. Distance from an external point to a line through the origin of the coordinates

Consider the line $l := X = O + \alpha\vec{u}$ and the point $F := \vec{f}$. Choose any two planes π_1 and π_2 such that $l = \pi_1 \cap \pi_2$. We may select these two planes in the following way [8]:

(1) Consider the plane π given by $\vec{x} \bullet \vec{u} = 0$.

(2) Choose, on the line l, the points O and M.

(3) Choose, on the plane π, the points R and T.

We get the planes

$$\pi_1 := \det\begin{bmatrix} x_1 & x_2 & x_3 \\ m_1 & m_2 & m_3 \\ r_1 & r_2 & r_3 \end{bmatrix} = 0, \quad \pi_2 := \det\begin{bmatrix} x_1 & x_2 & x_3 \\ m_1 & m_2 & m_3 \\ t_1 & t_2 & t_3 \end{bmatrix} = 0,$$

with $l = \pi_1 \cap \pi_2$.

Now we are in a position to apply to this concrete case the Anderson-Duffin formula

$$P_l(\vec{f}) := P_{\pi_1 \cap \pi_2}(\vec{f}) = 2(P_{\pi_1}(P_{\pi_1} + P_{\pi_2})^{\dagger} P_{\pi_2})(\vec{f}),$$

where

$$P_{\pi_1} = M_1(M_1^T M_1)^{-1} M_1^T,$$

$$P_{\pi_2} = M_2(M_2^T M_2)^{-1} M_2^T,$$

$$M_1 = [\vec{u}_1 \quad \vec{v}_1], \text{ with } (\vec{u}_1, \vec{v}_1) \text{ basis of } \pi_1$$

and

$$M_2 = [\vec{u}_2 \quad \vec{v}_2], \text{ with } (\vec{u}_2, \vec{v}_2) \text{ basis of } \pi_2.$$

The projection point is

$$\vec{s} := P_l(\vec{f}_{d(F,l)}),$$

where the distance from the point $F := \vec{f}$ to the line l is $d(F, l) = \| \vec{s} \|$.

2.2. General case: distance from an external point to a line

In this subsection, we are given a line l not containing the origin and an external point $F := \vec{f}$.

We look for the projection $P_l(\vec{f})$ of the point $F := \vec{f}$ onto the line l.

We use the Euclidean distance, so translations are allowed. We have to do a suitable translation, in order to apply what we have done in Subsection 2.1, and then we perform the reverse translation for returning back to the original situation.

We have the pair (F, l) formed by the point $F := \vec{f}$ and the line l given by $X = M + \alpha\vec{u}, \alpha \in \mathbb{R}$, with $F \notin l$.

For the sake of using the Anderson-Duffin formula, we must displace the line l to the origin of the coordinates.

We make the translation by the vector $\overrightarrow{MO} = O - M := -M := -\vec{m}$. So the pair (F', l') has to be considered, where $F' = F - M := \vec{f}' = \vec{f} - \vec{m}$ and $l' := O + \alpha\vec{u}, \alpha \in \mathbb{R}$.

We have, regarding distances, $d(F, l) = d(F', l')$.

Now we have

$$\vec{s}' := P_{l'}(\vec{f}') = P_{l'}(\vec{f} - \vec{m}) = P_{\pi'_1 \cap \pi'_2}(\vec{f} - \vec{m})$$

$$= 2(P_{\pi'_1}(P_{\pi'_1} + P_{\pi'_2})^{\dagger} P_{\pi'_2})(\vec{f} - \vec{m}),$$

where the planes π'_1 and π'_2 are constructed following the method presented in Subsection 2.1.

For the distance, we get $d(F, l) = d(F', l') = \| \vec{s}' \|$.

Returning back to the original situation, we reverse the translation by the vector $\vec{m}$. We obtain

$$\vec{s} := P_l(\vec{f}\,) = \vec{s}\,' + \vec{m}.$$

For the distance, we get $d(F, l) = \| \vec{s} \|$.

3. Euclidean Distance between Two Skew Lines

In this section, we construct the best approximation pair of two skew lines and consequently the distance between these lines.

We replicate the procedure we proposed in Subsection 2.2, in the sense that we use twice the translation and the corresponding reverse, one for each line. The external point here is the generic point of each line under consideration. For this reason, each best approximation point depends on parameters, this leading, in the final stage, to solving an overdetermined consistent system of twelve equations and two unknowns.

Let us present our four stages procedure.

Let us consider the skew lines l_1 and l_2 given, respectively, by

$$X = P + \alpha\vec{u}, \quad X = Q + \beta\vec{v}, \quad \alpha, \beta \in \mathbb{R}.$$

We search the projection point S_1 onto l_1 and the projection point S_2 onto l_2. Then we form a vector which achieves the distance $d(l_1, l_2)$ between the two lines, for instance the vector $\overrightarrow{S_1S_2}$ and, finally, we calculate the distance $d(l_1, l_2) = \| \overrightarrow{S_1S_2} \|$.

We have the following steps:

(1) Translation of the line l_1 to the origin of the coordinates. We translate the pair (l_1, l_2), so obtaining the pair (l_1', l_2'), where

$$l_1' := X' = P - P + \alpha\vec{u} = O + \alpha\vec{u}, \quad \alpha \in \mathbb{R}$$

and

$$l_2' := X' = Q - P + \beta\vec{v} := Q' + \beta\vec{v}, \quad \beta \in \mathbb{R}.$$

We have to consider the generic (current) point $Q_2'(\beta)$, on the line l_2', which is

$$Q_2'(\beta) = (q_1 - p_1 + \beta v_1,\ q_2 - p_2 + \beta v_2,\ q_3 - p_3 + \beta v_3) := \vec{q}_2'(\beta).$$

We need to find the projection $S_1'(\beta) := \vec{s}_1'(\beta)$ of the point $Q_2'(\beta) := \vec{q}_2'(\beta)$ onto the line l_1' which contains the origin.

So we are in a condition for using the Anderson-Duffin formula the way we did in Subsection 2.2.

We have

$$\begin{aligned}\vec{s}_1'(\beta) &= P_{l_1'}(\vec{q}_2'(\beta))\\ &= P_{\pi_1' \cap \pi_2'}(\vec{q}_2'(\beta) - \vec{m})\\ &= 2(P_{\pi_1'}(P_{\pi_1'} + P_{\pi_2'})^{\dagger} P_{\pi_2'})(\vec{q}_2'(\beta) - \vec{m}),\end{aligned}$$

where the planes π_1' and π_2', belonging to the pencil of planes through the line l_1', are constructed by the method used in subsection 2.1.

(2) Translation of the line l_2 to the origin of the coordinates. By translating the line l_2 to the origin, the pair (l_1', l_2') turns into the pair (l_1'', l_2''), where

$$l_2'' := X'' = Q - Q + \beta\vec{v} := O + \beta\vec{v}, \quad \beta \in \mathbb{R}$$

and

$$l_1'' := X'' = P - Q + \alpha\vec{u} := P'' + \alpha\vec{u}, \quad \alpha \in \mathbb{R}.$$

The current point on the line l_1'' is

$$P_1''(\alpha) = (p_1 - q_1 + \alpha u_1,\ p_2 - q_2 + \alpha u_2,\ p_3 - q_3 + \alpha u_3) := \vec{p}_1''(\alpha).$$

Regarding the projection onto l_2'', we have

$$\begin{aligned}\vec{s}_2''(\alpha) &= P_{l_2''}(\vec{p}_1''(\alpha))\\ &= P_{l_2''}(\vec{p}_1''(\alpha) - \vec{q})\end{aligned}$$

$$= P_{\pi_1'' \cap \pi_2''}(\vec{p}_1''(\alpha) - \vec{q})$$

$$= 2(P_{\pi_1''}(P_{\pi_1''} + P_{\pi_2''})^{\dagger} P_{\pi_2''})(\vec{p}_1''(\alpha) - \vec{q}),$$

where the planes π_1'' and π_2'' concerning the line l_2'' are constructed the way as in Subsection 2.1.

(3) Performing the reverse translation. Turning back the pairs (l_1', l_2') and (l_1'', l_2'') to the original pair (l_1, l_2), we obtain

$$S_1(\beta) = S_1'(\beta) + P, \tag{1}$$

onto the line l_1,

$$S_2(\alpha) = S_2''(\alpha) + Q, \tag{2}$$

onto the line l_2,

$$P(\alpha) = P_1''(\alpha) + Q, \tag{3}$$

onto the line l_1 and

$$Q(\beta) = Q_2'(\beta) + P, \tag{4}$$

onto the line l_2.

The linear system formed by equations (1), (2), (3) and (4) has twelve equations and two unknowns α and β. It is consistent for geometric reasons. Once solved, we get

$$\alpha = \alpha^*, \quad \beta = \beta^*.$$

(4) Final step. After the resolution of the above system (1), (2), (3) and (4), we obtain

$$S_1 = S_1(\beta^*), \quad S_2 = S_2(\alpha^*).$$

The best approximation pair (S_1, S_2) gives

$$\overrightarrow{S_1 S_2} = S_2 - S_1$$

and

$$d(l_1, l_2) = \| S_2 - S_1 \|.$$

(Geometrical considerations permit to write also $S_1 = P(\alpha^*)$ and $S_2 = Q(\beta^*)$.)

4. Appendix

The main objective of this Appendix is to present some known facts scattered in papers and books, in order that scientists, engineers, didacticians of mathematics and applied mathematicians may go further when dealing with delicate questions concerning mainly infinite dimensional spaces. We treat some aspects of: existence, computations, continuity of the generalized Moore-Penrose inverse; existence, computations, characterization of the linearity of projectors; properties of continuous operators; invariance of distances under translations; distance between two sets.

A1

Our main tool in the present paper is the Moore-Penrose inverse of a sum of two projectors onto two subspaces **M** and **N**.

What about the existence of such Moore-Penrose inverse?

In fact, the existence of $(P_{\mathbf{M}} + P_{\mathbf{N}})^{\dagger}$ presupposes that the subspaces **M** and **N** are closed and, furthermore, that the sum $\mathbf{M} + \mathbf{N}$ be closed as well [2].

(When we write closed subspace (of a normed linear space), the word closed is used in the sense of topology and not just in the sense of the algebraic operations).

In finite dimensional spaces, no problem does arise. In the infinite dimension context, things are more complicated, due to the fact that even if **M** and **N** are closed subspaces, the sum $\mathbf{M} + \mathbf{N}$ need not to be closed ([13, Introduction, p. 28, Section 15] and [3, Section 17, Problem 1.5.1]).

For $\mathbf{M} + \mathbf{N}$ to be closed when **M** and **N** are closed subspaces, see [24] when the orthogonality of **M** and **N** suffices; for other conditions, see [2], [3, Section 17, Problem 1.5.1], [4, p. 334, Ex. 1], [11, p. 226, Theorem 7.56], [5, p. 1424, Example 3.2]. See also [10, p. 199, Remark; p. 223, Theorem 9.35], where the conditions for $\mathbf{M} + \mathbf{N}$ to be closed are linked to the rate of convergence of the method of alternating projections.

The Moore-Penrose inverse is not continuous ([6] and [18]). It is a nice tool for theoretical results, but it is not computationally stable [18]. For details on computational aspects of the Moore-Penrose inverse, see [25].

A2

In this paper, we used projectors which were computed in an easy way. In approximation theory, in order to apply the gradient projection method, the calculation of the projection $R_C - C$ is a convex set is needed. This calculation might be not easy [11, p. 198].

The best approximation is unique in Euclidean spaces [12, p. 221]. The unicity of best approximation allows one to present an operator of best approximation, a projector, which, in general, is non-linear [12, p. 132].

Projector onto subspaces is linear (Hiriart-Urruty and Lemaréchal [14] asserts and Laurent [19] proves).

For non-linear projectors, see [4, p. 132, Theorem 11; Φ - metric projector].

Projectors, even if non-linear, are continuous (see [11, p. 184, Theorem 6.9] and [19, p. 44, Proposition 2.2.4]).

Projections onto the intersection of subspaces are a useful tool when studying the sum of subspaces [5].

A3

As we already said, the projector is continuous even when not linear.

Linear (affine) applications are (Lipschitz) continuous [26, pp. 44-45] where the Frobenius norm f of a matrix is the Lipschitz constant.

In infinite dimensional spaces, there are linear mappings that are not continuous ([22, p. 105] and [23, p. 146]).

A4

Here we deal with distances: distance from a point to a set and distance between two sets.

Sometimes there exists in writing and in speaking some confusion about metric and distance, though, in our opinion, not so grave, because the same happens with the concepts of vector product, inner product, determinant, etc., where the function and its value are given the same name. In order to get records straight, we spend some lines for this end [7, pp. 60-61].

A metric space is a pair consisting of a set *E* and a mapping

$$d : E \times E \to \mathbb{R}_+$$

$$(\vec{x}, \vec{y}) \to d(\vec{x}, \vec{y}),$$

satisfying the following properties:

(1) $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$;

(2) $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$ (symmetry);

(3) $d(\vec{x}, \vec{y}) \leq d(\vec{x}, \vec{z}) + d(\vec{z}, \vec{y})$ (triangle inequality).

The function *d* is called a *metric* and its value $d(\vec{x}, \vec{y})$, at $(\vec{x}, \vec{y})$, is called the *distance* between the points $\vec{x}$ and $\vec{y}$.

Note that a metric on a set *E* is a function defined, not on *E* but on E^2.

Now we consider the distances point/set and set/set ([26, p. 45] and [7, p. 67]).

Let *A* and *B* be subsets of a metric space endowed with the metric *d*. The distance between the two sets *A* and *B* is defined as the infimum $d(A, B)$ of the distances $d(\vec{a}, \vec{b})$, where $\vec{a} \in A$ and $\vec{b} \in B$:

$$d(A, B) = \inf\{d(\vec{a}, \vec{b}) : \vec{a} \in A, \vec{b} \in B\} = \inf\{\| \vec{a} - \vec{b} \| : \vec{a} \in A, \vec{b} \in B\},$$

where $\| \cdot \|$ stands for some norm in $\mathbb{R}^n$.

In particular, we have the distance from a point to a set. For every $\vec{x} \in E$, the distance from $\vec{x}$ to *B* is defined as the number

$d(\vec{x}, B) = d(\{\vec{x}\}, B) = \inf\{d(\vec{x}, \vec{b}) : \vec{b} \in B\} = \inf\{\| \vec{x} - \vec{b} \| : \vec{b} \in B\}.$

Some cautions and remarks are adequate.

• Regarding the distance point/set [26, p. 45]. In general, the *inf* in the definition of $d(\vec{x}, B)$ cannot be replaced with *min*: for example consider the punctured unit disk

$$B = \{\vec{x} : \| \vec{x} - \vec{0} \| = \| \vec{x} \| \leq 1, \vec{x} \neq \vec{0}\}$$

and note that $d(\vec{0}, B) = 0.$ However, [26, p. 45, Theorem 1.9.1], for a non-empty closed set *B*, *inf* can be replaced with *min*.

• Regarding the distance set/set [7, p. 67]

- The distance is not a metric. Despite its name, the distance $d(A, B)$ between the two sets *A* and *B* is not a metric on the set of subsets of *E*, for the triangle inequality is not satisfied. For example [7, p. 67], in $\mathbb{R}$, let $A = [0, 1], B = [1, 2], C = [2, 3].$ Then $d(A, B) = d(B, C) = 0$ while $d(A, C) = 1.$

- $d(A, B) = 0$ does not imply $A \cap B = \emptyset.$ For disjoint sets *A* and *B*, in general, $d(A, B)$ is positive. However, $d(A, B) = 0$ does not imply $A \cap B = \emptyset,$ that is, one may have $d(A, B) = 0$ when *A* and *B* are disjoint. We list two examples where the lack of boundedness is crucial.

∗ The sets

$$A = \{(x, y) \in \mathbb{R}^2 : xy \geq -1, x < 0\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2 : xy \geq 1, x > 0\}$$

are both closed and disjoint.

We have, however, $d(A, B) = 0$ [21, p. 115, Example 5.1].

∗ The distance from a branch of a hyperbole to one of its asymptotes is zero [7, p. 67, Example 3].

Note in both cases the strength of the hyperbole!

- On the invariance of distances under translations. In general, the metric is not translation invariant. However, [20, p. 37], given a norm $\|\cdot\|$ in a linear space X over $\mathbb{R}$ or $\mathbb{C}$, with the aid of a norm, we can introduce a metric in X, by defining the distance of two points in the following way: $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$. Every metric in a linear space that obeys the properties of invariance under translation and scalar multiplication:

$$d(\vec{x} + \vec{z}, \vec{y} + \vec{z}) = d(\vec{x}, \vec{y}), \quad d(\alpha\vec{x}, \alpha\vec{y}) = |\alpha| d(\vec{x}, \vec{y})$$

comes from a norm via $\|\vec{x}\| = d(\vec{x}, \vec{0})$. From the preceding lines, we conclude that every normed space is a metric space and that some metric spaces are normed spaces.

5. Conclusions and Remarks

By the adequate use of translations, an Anderson-Duffin formula was used to find the projection of a point onto a line not passing through the origin. The consideration of such a line as the intersection of two planes begs the need for generalizing the Anderson-Duffin formula from the intersection of subspaces to the intersection of linear varieties.

The Anderson-Duffin formula gives the projection onto the intersection of two subspaces through the Moore-Penrose inverse of the sum of two matrices which are the projections onto the referred to subspaces.

Engineers, scientists, didacticians of mathematics do feel comfortable as we work on spaces of finite dimension and use projections onto subspaces. In infinite dimensional spaces and regarding projections onto general sets, things are more complicated as shown in the Appendix.

Our approach was constructive and had didactical preoccupations. Extensions for higher dimension finite dimensional spaces are not straightforward and some cautions in more general contexts are needed. The Appendix is intended to guide more abstract minded readers.

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