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ABSTRACT

By determining the force of attraction that a spherical mass exerts on a mass point Newton succeeded in proving that gravitation is the reason for both the motion of the moon around the earth (and the planets around the sun) and for bodies falling down to the ground. In order to describe these motions in classical physics it was necessary to understand space, time, and matter such that these concepts could be quantified in a way that made an analysis possible by the methods of mathematics; in this development both new mathematical concepts were initiated as well as existing ones extended.

§1. Introduction. Survey of the talk

The first topic in this lecture will be a theorem from Newton's Principia Mathematica which is considered to be the central argument of this treatise: Newton determines the gravitational force that a spherical body exerts on a point outside of it. With this result Newton showed in particular that the same gravitational law governs the motion of the moon around the earth as well as the motion of a material body that falls down towards the surface of the earth.

When we ask under what assumptions such a statement is possible we recall how motion was understood in Antiquity. Then we present in particular Aristotle's concept of quantity that is basic for any description of physical phenomena by mathematical methods. Archimedes' law of the lever is an example from Antiquity that indicates how this concept was used; then we show in what sense it was extended to make modern physics possible.

§2. Newton's Principia Mathematica: The Central Argument

(i) Formulation of the Theorems: In the *Philosophiae Naturalis Principia Mathematica* by Isaac Newton, we find in Book I (*The Motion of Bodies*), Chapter XII ("The Attractive Forces of Spherical Bodies") the following statements:

> *"Proposition LXX. Theorem XXX: If toward each of the separate points of a spherical surface there tend equal centripetal forces as the squares of the distances from the point, I say that a corpuscle placed inside the surface will not be attracted by these forces in any direction."*

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> *"Proposition LXXI. Theorem XXXI: With the same conditions being supposed as in Prop. LXX, I say that a corpuscle placed outside the spherical surface is attracted to the center of the sphere by a force inversely proportional to the square of its distance from that same center."*

(ii) The Central Argument: When we call these results (and others that follow from them) central argument then there must be a good reason to do so. Newton himself supports this claim as his letter to E. Halley from June 20, 1686 shows: "I never extended y^e duplicate proportion lower than to y^e superficies of y^e earth & before a certain demonstration I found y^e last year have suspected it did not reach accurately enough down so low... There is so strong an objection against y^e accurateness of this proportion, y^t without my Demonstrations... it cannot be beleived by a judicious Philosopher to be any where accurate."1) The Principia appeared in 1687, so these results were found rather late.

The law of the duplicate proportion means that the attraction $\mathcal{F}(R)$ of a mass point at position R towards a mass at S is directed towards S, and its size equals.

$$\mathcal{F}(\mathrm{R}) = \frac{c}{RS^2}$$

Based on this law Newton could determine the path that a planet takes around the sun at position S, e.g. one focal point of an ellipse. Because the distances between the sun and a planet (or the earth and the moon) are very large compared to their diameters it is reasonable to consider these as mass points.

But if one now considers a body of centimeter size about 1m above the ground here on earth, and if one assumes that every point of the earth exerts a force of the size above, then it is by no means obvious

Fig. 1. Isaac Newton

that all this results in the rather simple statement that the force is porportional to the distance of the material body from the center of the earth to the power of (-2). And it was this point that Newton didn't understand initially, namely that the law could be applied to points of the earth that are deep under the surface. But as soon as there are demonstrations, i.e. mathematical theorems, the question is settled, regardless what some "judicious Philosopher" might believe.

There is another reason for calling these propositions the central argument. In the portrait of Isaac Newton, painted towards the end of his life by "the studio of Enoch Seeman"[2], the light shines on Newton, on the terrestrial globe in the back and on the book that lies on the desk. This is the third edition of the Principia Mathematica from 1727, opened up on pp. 204-205, where consequences of Theorems XXX and XXXI are discussed. It is quite likely that this is not a coincidence, so it stresses the importance of Newton's results about the gravitational force of a spherical body.

(iii) Proof of Theorem XXXI:

It is not possible to present a detailed proof in this lecture, but I will hint at some characteristic points of the procedure. The figures that belong to Theorem XXXI are really striking because the spherical body is shown twice, and the distances of the points P and p, where the force exerted by the spherical body is to be determined, are different. That already reminds us on proofs from classical Greek geometry; when the area $\mathcal{A}$(R) of a disc of radius R was to be determined, the statement would read like

$$\mathcal{A}(\mathrm{R}) : \mathcal{A}(\mathrm{r}) = R^2 : r^2,$$

thus avoiding the quantity π, the area of the unit disc, which could not be determined.

Now consider the lines PK and PL that determine the arcs *a(HI)* and *a(KL)* of the circle; rotation of *a(HI)* about the axis PB gives the spherical ring $\mathcal{R}$*(HI)*. Newton compares the contribution of $\mathcal{R}$*(HI)* with the corresponding one in the second configuration, where the point, now called p lies closer to the sphere. We draw lines pk and pl such that the corresponding arcs *a(hi)* and *a(kl)* have the same lengths as *a(HI) and a(KL).* This construction and many other steps can be done by using classical geometry. A detailed proof is given by Dana Densmore in her book *Newton's Principia: The Central Argument*[3], where in particular the theorems from Euclid's elements are quoted that Newton used.

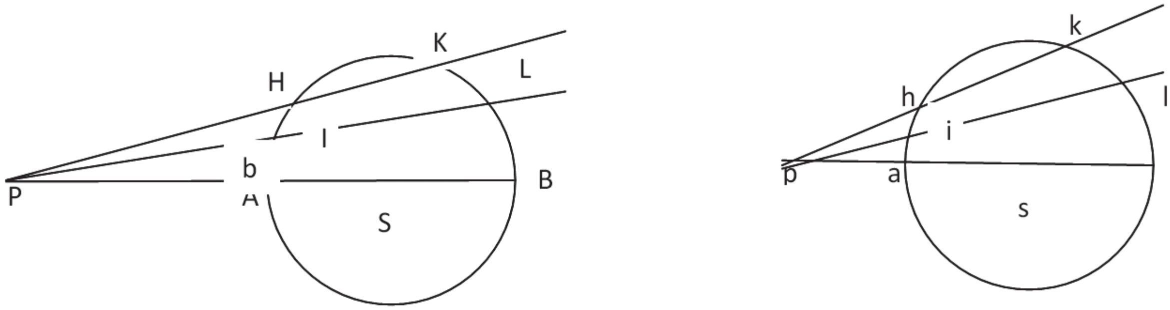

Fig. 2. Proof of Theorem XXXI

So eventually Newton shows that the force $\mathcal{F}(\mathrm{P})$, acting on a mass point at P, or p, resp. satisfies

$$\mathcal{F}(\mathrm{P}) \; : \; \mathcal{F}(\mathrm{p}) \; = \; ps^2 \; : \; PS^2;$$

This is a comparison of the forces acting on different points rather than an equation like

$$\mathcal{F}(\mathrm{P}) = \frac{c}{PS^2},$$

as we would write nowadays.

There are variants of Newton's proof that give the same result but are closer to modern arguments, e.g. by J. E. Littlewood[4)], who characterizes Newton's original reasoning by the statement: "The proof leaves the reader in helpless wonder." This may be taken as an invitation to study this beautiful piece of mathematics line by line.

§3. Motion in Classical Greek Philosophy and the Category of Quantity (Aristotle)

Newton's theorems are the basis for describing the motion of the planets around the sun and of the motion of a material body here on earth. In order to investigate our concept of nature that makes this description possible, we briefly consider motion as it was understood in classical antiquity.

(i) Motion in Aristotle's Physics

Neither in the work of Aristotle nor of any other philosopher from Antiquity or the Middle Ages has motion of material bodies been analyzed in a satisfactory way, but comparing our concepts with the classical ones (from which they developed) helps to understand them better. With this aim in mind I shall not concentrate on the deficiencies of ancient physics.

In the first sentence of *Physics*, Book III, Aristotle states that the phenomena in nature cannot be understood, if motion is not understood:

"Nature is the underlying principle of motion and change, and it is the object of our inquiry. We must therefore see that we understand the meaning of "motion"; for if it were unknown the meaning of "nature" too would be unknown."[5)]

This statement was at the time of Aristotle far from trivial, because there were other philosophers who considered change to be an illusion, and they held that knowledge must aim at ideas that do not change at all. From Aristotle on, it was more and more accepted that motion is the truly basic concept of nature, as Albert of Saxony (1316-1396) states: "Ignorato motu ignoratur natura."

And the next sentences in *Physics*, Book III, give already the connection with mathematics:

"When we have determined the nature of motion our next task will be to attack in the same way the terms which are involved in it. Now motion is supposed to belong to the class of things which are continuous; and the infinite presents itself first in the continuous – that is how it comes about that "infinite" is often used in definitions of the continuous (what is infinitely divisible is continuous). Besides these, place, void, and time are thought to be necessary conditions of motion."[6)]

When we investigate the motion of material bodies we can ask for the causes that produce – and stop – it, and we can analyze quantitatively which means that we concentrate on how slow or fast a motion is.

(ii) Motion of heavenly bodies and motion on earth: With respect to the causes of motion I will state only one point. Aristotle distinguishes the motion of a body here on earth strictly from the motion of the sun, the moon, and the stars. A motion on earth starts at some point and at some well-defined time, and it ends at some point and time, both due to certain causes. (And then one may characterize the causes, as Aristotle does: causa materialis, finalis, efficiens, and formalis.) The motion of a heavenly body is of a completely different nature: it has no beginning and no end, because it proceeds in circles, which implies that it does not change; and only this type of motion is appropriate to its divine character. And because of this property they are not to be investigated as are things here on earth. The example of Anaxagoras (499-428) shows the radical difference between the divine and the profane region. He claimed that the sun is a hugh, hot mass, and that the moon is cold mass that does not shine by itself but reflects the light from the sun. Because of such statements he was accused in 430 of ἀσέβεια7): he was not accused of making some incorrect statement; because of their divine character the sun and the moon are not objects of our investigation at all – this was considered committing a sacrilege –, hence we have to face them with respect and devotion. Anaxagoras was sentenced to death; his friend Perikles could arrange that he was safed from this punishment, but he had to go into exile.

It was monotheism that changed this attitude, as we can see from the biblical story of the creation in Gen 1,1 – 2,4a which was written around 500 BC. Israel at that time was dominated by the Babylonians, and for them Isis, the goddess of the moon, was the highest deity. Now Gen 1,14-19 states that God created a big lamp and a small lamp, so the deities of the sun and the moon are not even mentioned by name, and they are created for certain purpuses, in particular to indicate the years and the holy days of obligation, when Israel should celebrate their god, because the calender in Israel is determined by the phases of the moon. Clearly an object with such a purpuse can be investigated like any other object here on earth.

§3 (iii) The Category «how much» (ποσόν): In the quotation from *Physics*, Book III, above, Aristotle speaks of "continuity", a mathematical concept, at which we now look in more detail.

"Quantity is either discrete or continuous. (...) Instances of discrete quantities are number and speech; of continuous lines, surfaces solids, and, besides these time and place.

In the case of the parts of a number, there is no common boundary at which they join. For example: two fives make ten, but the two fives have no common boundary but are separate. (...) Number, therefore is a discrete quantity.

*A line, on the other hand, is a continuous quantity, for it is possible to find a common boundary at which its parts join. In the case of the line, this common boundary is the point; in the case of the plane it is the line: for the parts of the plane have also a common boundary."*8)

*"Quantum" means that which is divisible into more constituent parts of which each is by nature a "one" and a "this". A quantum is a plurality (multitude) if it is numerable, a magnitude (quantity), if it is measurable. "Plurality" means that which is divisible potentially into non-continuous part, "magnitude" that which is divisible into continuous parts; of magnitude, that which is continuous in one dimension is lenght; in two breath, in three depth. Of these, limited plurality is number, limited lenght is a line, breath a surface, depth a solid."*9)

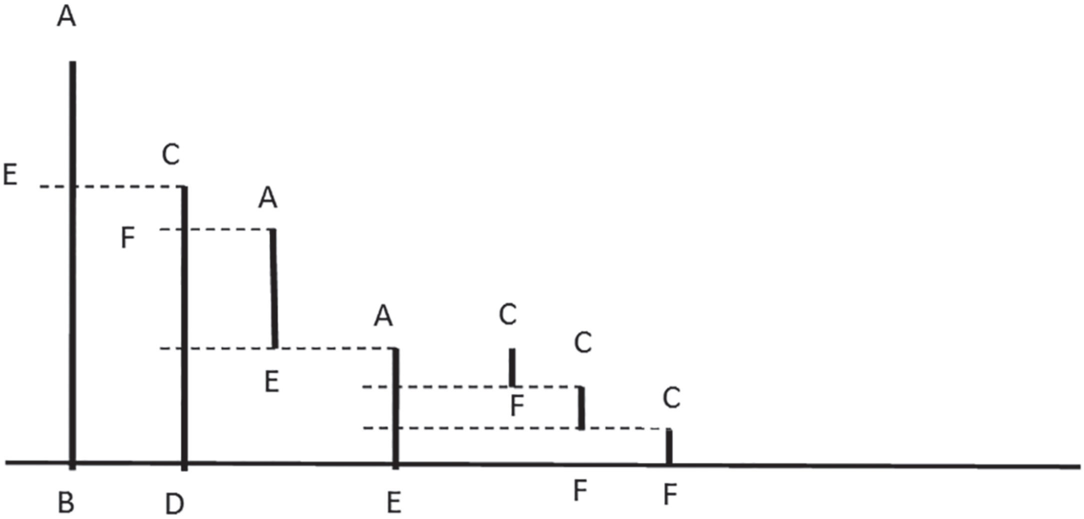

Fig. 3. Method of Subtracting in Turn.

For us the distinction between multitude and quantity is evident, but in Antiquity the occurrence of incommensurate lines was a truly sensational event; therefore I will briefly recall some aspects of this development. The greek geometers had found a method, called subtracting in turn, to compare two lines AB and CD, essentially by counting. We subtract the shorter line CD from the longer one AB as often as possible; here we can subtract it once, because the rest AE is shorter than CD. Then we subtract AE from CD two times; that leaves the rest CF. And CF fits three times into AE, and the procedure stops. So we have shown CD = 7 · CF and AB = 10 · CF, which means that the lines have a common measure CF, and their length satisfy AB : CD = 10 : 7.

When we now ask for a common measure of certain lines like a side and a diagonal in a regular polygon we may proceed in the same way, but now these lines as well as their differences are part of a geometric figure. Hippasos of Metapont (ca. 510-440) found in this case for the side s=AB and the diagonal d=BD that the difference

d_1:= d-s is the length of the diagonal of the inner, smaller pentagon, and s_1:= s-d_1 is its side.[10] Therefore the process of subtracting in turn does not stop, it gives in every step the same relation for the side and the diagonal of a smaller pentagon, and there cannot exist a common measure for the side and the diagonal of a pentagon.

The same phenomenon occurs in the square, and here we can give in addition to a geometric proof like the one above also an algebraic one that was also well known in classical times. In the unit square the length of the diagonal equals $\sqrt{2}$, so if the diagonal and the side of length 1 are commensurate, there must hold

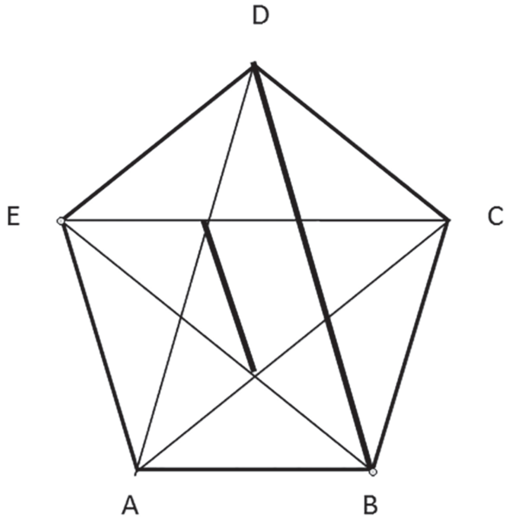

Fig. 4. Regular pentagon.

$$\sqrt{2} = \frac{p}{q},$$

for some numbers p, q$\in \mathbb{N}$; we may assume that p and q have no common divisor. Squaring this equation gives

$$2 \cdot q^2 = p^2,$$

hence p is even: p = 2r for some r. If we insert this into the last equation we obtain

$$2 \cdot q^2 = 4 \cdot r^2,$$

so q is an even number, too, which is a contradiction to our assumption that p and q have no common divisor.

Incommensurability was definitely a phenomenon that made a lasting impression on everyone in Antiquity, not just on the Pythagoreans who claimed that the world is ordered according to number. We present some examples that show how Aristotle referred to this item.

"For all who affect an argument per impossible infer syllogistically what is false, and prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory; e.g. that the diagonal of the square is incommensurate with its side, because odd numbers are equal to evens if it is supposed to be commensurate."11)

Aristotle discusses the reductio ad absurdum, and as an example he chooses to the proof that we just presented, namely that $\sqrt{2}$ cannot be equal to a fraction of natural numbers. But he refers to the punch line only, namely that a number cannot be both even and odd – this is not to be expected in a text that is part of the Organon, which contains introductory topics of Philosophy.

The Metaphysics of Aristotle starts with the statement "All men by nature desire to know."12), and they do so for the following reason: "For it is owing to their wonder that men both now begin and at first began to philosophize."13) Here wondering means that we do not understand why things are as they are.

"For all men begin, as we said, by wondering that things are as they are, as they do about self-moving marionettes, or about the solstices or the incommensurability of the diagonal of a sqare with its side; for it seems wonderful for all who have not yet seen the reason, that there is a thing which cannot be measured even by the smallest unit. But we must end in the contrary and, according to the proverb, the better state, as is the case in these instances, too, when men learn the cause; for there is nothing that would surprise a geometer so much as if the diagonal turned out to be commensurate."14)

The examples that Aristotle presents are figures that can move by themselves, these are things that are made by man, then the solstice as a phenomenon that is investigated in astronomy, and eventually incommensurability, an example from theoretical knowledge, and this point is discussed much longer that the other two.

Aristotle uses the concept of quantity in his work "On the Heavens", e.g. when he proves that there cannot be a body of infinite weight in the universe15), but this does not lead to an equation involving several quantities which states some law of physics. This step was done first by Archimedes, who extends the concept of magnitude and uses it in a way that is still valid today.

§4. Quantity and the Law of the Lever (Archimedes)

In the work of Archimedes we find several contributions where the concept of quantity plays an essential role. He calculated the area of certain figures like the segment of the parabola, thus extending the quantity "area", which was first defined for elementary figures only, to more general situations.

(i) Area of the Segment of a Parabola

Consider a parabola $\mathcal{P}$(ACB) and define the segment $\mathcal{S}$(ABC) to be that part below the parabola which is bounded by the line AB

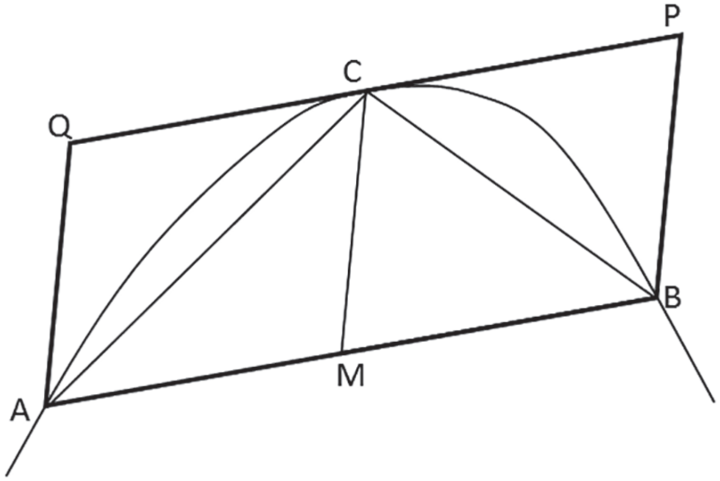

Fig. 5. Segment of a parabola.

The area of this segment equals the area of the triangle $\Delta(ABC)$ multiplied by $4/3$, where the point C of the parabola is chosen in such a way that the tangent in C is parallel to the segment AB:

$$|\mathcal{S}(ABC)| = 4/3 \cdot |\Delta(ABC)|.$$

The proof is truly ingeneous: The triangle $\Delta(ABC)$ defines two smaller segments of the parabola, determined by the sides AC and BC and in these parts one defines new triangles as before; each one has ⅛ of the area of $\Delta(ABC)$. If we continue in this way we get a geometric series:

$$1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots=\frac{4}{3}.$$

How does Archimedes prove this result? Today we would argue like this: the geometric series $\sum_0^\infty q^n$ is convergent for $|q|< 1$, and its limit is S = $\frac{1}{1-q}$, which gives ${}^4/_3$ for q=¼. Archimedes shows that S$>{}^4/_3$ as well as S $<{}^4/_3$ are impossible, hence there must hold equality; he does not indicate how he found this value.

In the „Works of Archimedes“16), edited by T.L. Heath, one finds the following drawing that is certainly an elegant geometric argument that gives the limit ${}^4/_3$.

From the unit square we remove the square with sides of length ½ and get the region A, whose area is ¾. If we proceed in this way in the smaller square we obtain the region B of size ¾-¼. Continuing in this way we get C, D, etc. whose area is always ¾ of the previous one. As A, B, C, D, (…) fill the unit square we have

$$1 = \frac{3}{4}\cdot\left(1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\dots\right) ,$$

which is the result. It must be noted, however, that this figure is not contained in the critical edition17) of the works of Archimedes; here and on many other places Heath goes his own way in editing and translating classical texts.

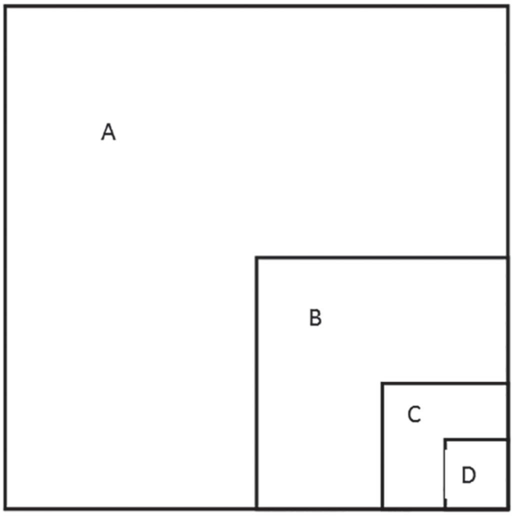

Fig. 6. Limit of the series.

(ii) The Law of the Lever: Archimedes determined the area of a segment of a parabola twice, the proof by exhaustion being the second one. His first proof is based on the law of the lever: Two weights A and B are in equilibrium in if their distances from the fulcrum Γ of the lever are inversely proportional.

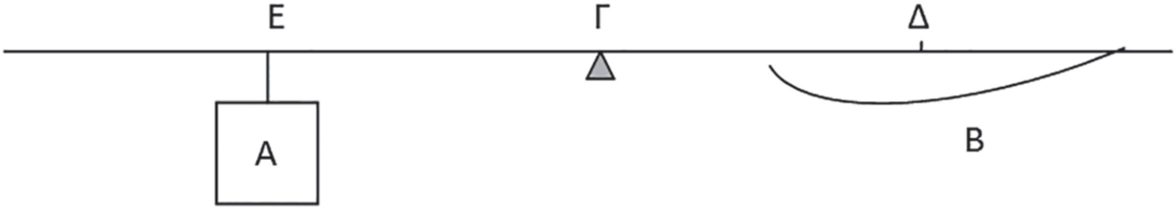

Fig. 7. Area determined by law of the lever

The first question is: How are the distances of the weights A and B to Γ defined? Distances are defined between points, hence one must associate to the weights A, e.g. a square, and B, e.g. a segment of a parabola, two points – and this is done by introducing the center of gravity: the weights A and B are replaced by points E and Δ that are supposed to have the same weight than the square and the segment.

$$A : B = \Gamma\Delta : \Gamma E$$

The proof shows how Archimedes works with this concept, but it is too long to be included here.

The main point that makes this mathematical formulation of a physical law possible is the concept of the center of gravity. This abstraction allows for introducing a quantity, namely a distance between two points; one then can formulate proportions between these quantities and finally write the law of the lever as a mathematical equation.

This is another example that shows how Archimedes extended the concept of quantity, and this procedure became later the model for formulating physical laws in mathematical terms.

§5. Quantity in Newton's Principia Mathematica

(i) Definitions I – V: Eduard Jan Dijksterhuis, historian of Science from Utrecht (who worked for a long time as a teacher at a high school in Tilburg) once suggested that we should memorize the first sentences of the important books of Science as we did in school with the works of authors like Caesar and Homer. So let's look at the first sentences in Newton's Principia:

„Quantitas materiae est mensura ejusdem orta ex illius densitate et magnitudine conjunctim."18)

In the following line he explains by a very elementary example that the quantity of matter is the product of the density and the volume of a material body. What is new here with respect to the examples from the work of Archimedes, will become clear when we look at the second definition.

„Quantitas motus est mensura ejusdem orta ex velocitate et quantitate conjunctim."19)

The concept of quantity has been developed over the centuries, in particular during the Middle Ages, such that in Newton's time a quantity can be defined as a product of other quantities. The difference to Archimedes' use of quantities is significant: now one can define the product of the weight A and its distance ΓE from the fulcrum as a new quantity $\mathrm{A} \cdot \Gamma\mathrm{E}$, which we call the momentum of the weight, and doing the same for the weight B, we can now formulate the law of the lever in a very elegant way:

$$\mathrm{A} \cdot \Gamma\mathrm{E} = \mathrm{B} \cdot \Gamma\Delta.$$

So the notion of quantity, as we see it in Newton's definitions I and II is clearly a further step of abstraction which took a long way to do.

Definition III „The innate force of matter is a power of resisting, by which every body, as much as it has, continues in its present state, whether it be of rest, or of moving uniformly forward in a right line."20) Some lines later the innate force of matter is called „nomine significantissimo" the inertial force. The principle of inertia is probably the best topic to point out the fundamental difference in the concept of motion in modern science and in antiquity. The main point is that the acceleration of a mass is proportional to the exterior force acting on it, and consequently the state of rest and the uniform motion are equivalent. Therefore the uniform motion can now be called a state (status), a term that literally means the absence of motion. Aristotle defined motion to be the opposite of the state of rest, and each motion starts at some position and some time from rest – and it ends at some position and time.

Then in Definition IV „An impressed force is an action exerted upon a body, in order to change its state, whether of rest or of uniform motion in a straight line."21) Newton defines the forces that produce some motion, and he gives the following examples:

a) A shock may cause a motion.

b) The pressure exerted on an object by a moving fluid results in some motion of the object.

c) The centripetal force generates a motion.

The last one is defined in Definition V: „A centripetal force is that by which bodies are drawn or impelled or any way tend towards a point as to a center."22) Here he lists three forces, namely d) gravity by which bodies tend towards the center of the earth, e) the magnetic force, and f) the force that keeps the planets in their orbits rather than moving away on a straight line.

(ii) Concluding Remarks: Now we can come back to Newton's theorems and the Central Argument, as it was outlined in §2. The main step consists of determing the "impressed forces", in particular the centripetal forces in such a way that these quantities and the corresponding acceleration of bodies are coupled in mathematical equations from which the orbits of planets, comets or projectiles can be deduced. I tried to show some new properties of the description of processes in nature by modern means, in particular as far as the application of mathematics is concerned. In doing so, I was somewhat unfair to Newton because I stressed the mathematical proof of Theorem XXXI. For Newton this was not the end of an investigation but the starting point for studying much more general configurations like the attraction od non-spherical bodies, the shape of a gravitating body that rotates about an axis and is therefore flattened at the poles, the attraction of two spherical bodies etc. Later, in Book II, he investigates the motion in resisting media, and all these topics show that the applications to various physical problems are his main concern. When I entered this lecture hall here in the academy I saw a beautiful faience stating: *Nisi utile est quod facimus, stulta gloria*. In the sense of this proverb, Newton's glory was definitely superb, because he established a theory that is certainly very useful.[23)]

These new methods of investigation have replaced elder ones, and they changed the way we look at nature. This is the topic of an intense philosophical discussion, about which I cannot report here. But I would like to present an example by Newton himself about this philosophical background; it is taken from a letter to Roger Cotes, who worked with Newton when the second edition of the *Principia* was prepared which appeared in 1713.

"Indeed, I have not been able to deduce the reason of these properties of gravity from phenomena, & I do not feign hypotheses. For whatever is not deduced from phenomena is to be called an hypothesis; and I do not follow *Hypotheses*, whether Metaphysical or Physical whether of occult qualities or Mechanical. It is enough that gravity should really exist, & act according to the laws expounded by us, & should suffice for all motions of the celestial bodies & and of our sea."[24)]

According to ancient philosophy it was the quality of some material body that induced a motion that eventually led the body to its appropriate place. Newton rejects such a conception; it is enough to formulate the laws by certain quantities, and it is not necessary to follow some hypotheses. Here he probably thinks of a mechanical hypothesis like the one introduced by Descartes, who assumed that the space between the central body and the planets is filled with some material that eventually generates the motion of the planets. And Huygens, a strict Cartesian, calculated that these vortices move around the earth with 14-times the rotational velocity of the earth.

I would like to close with a short remark about the last word of the citation from Newton's letter; here, as well as in the Principia on various places, he says "our sea", when "the sea" would be appropriate. It was probably a common, but certainly not modest phrase in England at that time. Floris Cohen, a dutch historian of science has investigated the time of the scientific revolution in a broader context.[25)] He not only analyzes the new way we look at the processes in nature but also many other attitudes that have changed. What is our place here on earth? How do we use the resources? Clearly the new science was the basis for a great technical development that enabled big economical changes. And, according to Cohen, there must first be some idea that gives a new picture. As an example he refers to seafaring and asks: What is necessary to cross the ocean and to reach new continents? Navigation, building of ships, providing the means to live on a ship for a long time, and many more solutions to practical problems must be found, before the crossing of the ocean can be tried. But the truly revolutionary idea consists in a different view of the world, in setting new goals that will be pursued. And this was achieved first by the Portuguese nation, as we all know well. R. Hooykaas has shown that these Voyages of Discovery were an important step that made the scientific revolution possible.[26]

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12) Aristotle, Met.I.1.980a21.
13) Aristotle, Met.I.2.982b12-13.
14) Aristotle, Met.I.2.983a12-23.
15) Aristotle, De caelo I.6.273a7-274b30.
16) T.L. Heath, The Works of Archimedes, Mineola, NY, 2002 (Dover reprint of the edition from 1897, Cambridge).
17) I.L. Heiberg, Archimedes, Opera Omnia, vol. II, Stuttgart, 1972, p.310.
18) Isaac Newton's Principia Mathematica, ed. by A. Koyré and I.B. Cohen, vol. I, Cambridge, MA, 1972, p.39 (The quantity of matter is the measure of the same, arising from its density and bulk conjointly.).
19) ibid., p. 40 (The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.).
20) Sir Isaac Newton's Principia, Motte's Translation, revised by F. Cajori, vol. I, Berkeley, 1962, p.2.
21) ibid., p. 40.
22) ibid., pp. 2-3.
23) In Book II, Section VII of the Principia (The motion of fluids, and the resistance made to projected bodies) Newton determines the shape of least resistance for rotationally symmetric bodies of given length and base, thus solving a variational problem that is much harder to deal with than the one that J. Bernoulli proposed almost 10 years later, and which is regarded as the starting point of the modern calculus of variations. In the Scholium to Theorem XXVIII Newton says: „This proposition I conceive may be of use in the building of ships.", ibid. p.333
24) This is part of the Scholium that Newton included in his letter to Cotes from 2 March 1712/13, cf. The Correspondence of Isaac Newton, vol. V, Cambridge, 1975, pp.241f.; see also I.B. Cohen, Introduction to Newton's Principia, Cambridge, MA, 1971, pp. 241f.
25) H.F. Cohen, The Scientific Revolution. A Historiographical Inquiry, Chicago, 1978.
26) ibid., pp. 354-357, and the literature cited there.