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Research Article

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The explicit formula for Gauss-Jordan elimination applied to flexible systems

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Abstract: Flexible systems are obtained from systems of linear equations by adding to the elements of the coefficient matrix and the right-hand side scalar neutrices, which are convex groups of (non-standard) real numbers. The neutrices model imprecisions, giving rise to calculation rules extending informal error calculus. Stability conditions for flexible systems are given in terms of relative imprecision and size of determinants. We then apply the explicit formula for the elements of the successive intermediate matrices of the Gauss-Jordan elimination procedure to find the solution of flexible systems, keeping track of the error terms at every stage. The solution respects the original imprecisions in the right-hand side and is the same as the one given by Cramer's rule.

Keywords: Gauss-Jordan elimination, flexible systems, stability, external numbers

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1 Introduction

Flexible systems of linear equations were introduced in [1]. The coefficients and the right-hand side of these systems are external numbers of non-standard analysis, which can be seen as real numbers with small error terms. They have essentially the same properties as the real numbers, apart from some correcting terms that may appear due to the propagation of these errors along the calculations. A flexible system $\mathcal{A}|\mathcal{B}$ with coefficient matrix $\mathcal{A}$ and right-hand side $\mathcal{B}$ is defined as an inclusion, and its solution is the set S of all real vectors x such that $\mathcal{A}x \subseteq \mathcal{B}$.

When solving systems of linear equations numerically, to be reasonably successful it is important that imprecisions should not be too big and determinants should not be too small. In our approach, these observations are formalized by *stability conditions*, given for squared non-singular coefficient matrices $\mathcal{A}$. In particular, the relative imprecisions of elements of $\mathcal{A}$, when compared to $\Delta \equiv \det(\mathcal{A})$, should be at most of the same order as the relative imprecisions of the right-hand side $\mathcal{B}$, and Δ should not be so small, that dividing by it would increase the latter imprecision significantly. We derive that under the stability conditions presented in this article each Gauss-Jordan operation transforms a stable system into a stable system, with at most controlled growth of the imprecisions, obtaining at the end a system with coefficient matrix almost equal to the identity matrix, and a right-hand side with imprecisions not exceeding the original imprecisions; the right-hand side is then the solution set and is equal to the solution given by Cramer's rule. This is formalized in the following theorem.

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Theorem 1.1. (Main theorem). *Assume the system $\mathcal{A}|\mathcal{B}$ is reduced, non-singular, uniform, stable, and properly arranged with respect to a representative matrix P of $\mathcal{A}$. Let S be its solution. Then*

$$S = G = \mathcal{G}^P\mathcal{B} = \left(\frac{\det(M_1)}{\Delta}, \dots, \frac{\det(M_n)}{\Delta}\right)^T.$$

A system is said to be uniform if all components of the right-hand side $\mathcal{B}$ have the same imprecision. The Gauss-Jordan procedure is carried out with real numbers and written in matrix form, using a reduced representative matrix P of $\mathcal{A}$, i.e., a real matrix consisting of elements of the entries of $\mathcal{A}$; it is properly arranged in the sense that every pivot lies on the principal diagonal and is maximal within its row. The set $\mathcal{G}^P\mathcal{B}$ results from applying the Gauss-Jordan procedure to the right-hand side; in fact, this set does not depend on the choice of P, and we may denote it as well by G. For $1 \le i \le n$ the symbol M_i indicates the matrix obtained from $\mathcal{A}$ by substituting the ith column by the right-hand side $\mathcal{B}$.

There is an extensive literature on error analysis for the Gaussian and Gauss-Jordan elimination procedures, see, e.g., [2–6], which contain many more references. Often the approach is asymptotic, as a function of the number of variables n. Some key notions are the *growth factor* $\rho \equiv \frac{\max_{i,j,k} | a_{i,j}^{(k)} |}{\max_{i,j} | a_{i,j} |}$, where $k \le n$ and $[a_{i,j}^{(k)}]$ is the k-th intermediate matrix, and the *condition number* in the form of the product of norms $\operatorname{cond}(A) \equiv \|A\| \cdot \|A^{-1}\|$.

Here we choose an alternative asymptotic approach. We let n be standard, and suppose that the errors are convex subgroups of the real number system called (scalar) neutrices; typically, they are contained in the infinitesimals. So an external number is the sum of a real number and a neutrix, while the latter captures the intrinsic vagueness of an error term or an imprecision by the Sorites property of being invariant under some additions [7]. Important tools in our setting are explicit formulas for the elements of the intermediate matrices [8,9] and estimates of determinants and their principal minors; this seems somewhat natural, since by Cramer's rule the solution of linear systems is stated in the form of quotients of determinants, and the Gauss-Jordan operations are carried out with quotients of minors; we point out that there exists a relationship between the orders of magnitude of determinants and their principal minors.

Our approach to the error analysis of non-singular linear systems cannot be extended in a straigthforward way to singular systems, in particular systems with more variables than equations. Indeed, the stability conditions depend in a large part on properties of the determinant of the coefficient matrix and also Cramer's rule is no longer valid. An additional complication comes from the fact that Gaussian elimination will lead to rows in the coefficient matrix with neutrices instead of zeros, and further analysis is needed to decide whether these rows can be ignored.

We were inspired by Van der Corput's program of Ars Negligendi [10], where neutrices are defined as groups of functions; however, these neutrices do not allow for such a strong algebraic structure as the set of external numbers. An application of our approach may be seen to fall within Van der Corput's program: when a flexible system is stable, we may as well solve a simpler system, neglecting all terms in the coefficient matrix contained in its biggest neutrix.

This article has the following structure. In Section 2, we recall some basic properties of non-standard analysis, and of neutrices and external numbers. We also give some notions and notations for the Gauss-Jordan operations, matrices with external numbers and flexible systems. We distinguish various forms of solution sets, including a generalization of Cramer's rule, and indicate how they are related. We define the notion of stability, and state the principal tool for the proof of the Main theorem, saying that stability is respected by the Gauss-Jordan operations. Section 3 contains the proof of the Main theorem. In Section 3.1, we show the impact of each step of the Gauss-Jordan procedure on the size of the neutrices. Cramer's rule for flexible systems is proved in Section 3.2. In Section 3.3, we show how the solution sets are related. We prove the Main theorem in Section 3.4. In Section 4, we define equivalent systems, as systems having the same solutions, and show in the case of stability how a given system may be substituted by a simpler equivalent system.

2 Background

We start with some background on non-standard analysis, in Section 2.1. In Section 2.2, we recall the notions of neutrix and of external number used to model the imprecisions. In Section 2.3, we introduce some notions and notations with respect to the Gauss-Jordan operations. These will be carried out in the form of matrix multiplications, for which explicit expressions are given. Section 2.4 contains some terminology with respect to external matrices. In Section 2.5, we recall the definition of flexible systems of linear equations, introduce a notion of stability and state the principal auxiliary result to the Main theorem, saying that the Gauss-Jordan elimination procedure transforms a stable system into a stable system. In Section 2.6, we define solution sets and show how they are related, where equality holds under the condition of stability.

Notation 2.1. Throughout this article we use the symbol $\subseteq$ for inclusion and $\subset$ for strict inclusion.

2.1 Non-standard analysis

We use the axiomatic form of non-standard analysis Internal Set Theory (IST) of [11]. To the language $\{\in\}$ of common set theory ZFC a new predicate "standard" is added, denoted by "st"; an important feature is that, next to the standard numbers, infinitesimals and infinitely large numbers are already present within the ordinary set of real numbers $\mathbb{R}$. Formulas containing the symbol "st" are called *external*. External formulas are needed to define neutrices and external numbers, but formally these are not sets of IST. However, they are sets in the extension HST of a bounded form of IST given by Kanovei and Reeken in [12]. For introductions to IST we refer to e.g., [13,14] or [15]. An introduction to a weak form of non-standard analysis sufficient for a practical understanding of our approach is contained in [16]. An important tool is the principle of *external induction*, which states that induction is valid for all IST-formulas over the standard natural numbers. To be precise, if A is a formula in the language $\{\in, \text{st}\}$ such that $A(0)$ holds and $A(n) \Rightarrow A(n+1)$ holds for every standard $n \in \mathbb{N}$, then $A(n)$ holds for all standard $n \in \mathbb{N}$; in fact, mutatis mutandis, the base step of External induction may be $A(n_0)$, for any standard natural number n_0.

A real number is *limited* if it is bounded in absolute value by a standard natural number, and real numbers larger in absolute value than all limited numbers are called *unlimited*. Its reciprocals, together with 0, are called *infinitesimal*. *Appreciable* numbers are limited, but not infinitesimal. The set of limited numbers is denoted by $\pounds$, the set of infinitesimals by $\oslash$, the set of positive unlimited numbers by $\not\infty$ and the set of positive appreciable numbers by @; these are all external sets in the sense of HST, since they can be defined by an external formula in which the symbol "st" is bounded by $\mathbb{N}$.

2.2 Neutrices and external numbers

We adopt the terminology from [17], which also contains illustrative examples. More complete introductions to external numbers are contained in [16,18,19]; the latter contains a full list of axioms for the operations on the external numbers. We recall here some useful notions and results.

The *Minkowski operations* on subsets A, B of $\mathbb{R}$ are defined pointwise, e.g.,

$$A + B = \{a + b | x \in A, y \in B\},$$

where, with some abuse of language, the symbol + is used both for the addition of real numbers and of sets of real numbers.

Definition 2.2. A (*scalar*) *neutrix* is an additive convex subgroup of $\mathbb{R}$. An *external number* is the Minkowski-sum of a real number and a neutrix.

Each external number has the form $\alpha = a + A$, where A is called the *neutrix part* of α, denoted by $N(\alpha)$, and $a \in \mathbb{R}$ is called a *representative* of α. If $N(\alpha) = \{0\}$, the external number α may be identified with a real number a. We call α *neutricial* if $\alpha = N(\alpha)$ and *zeroless* if $0 \notin \alpha$. The property of being a neutrix is unbounded, for neutrices can be defined by a formula with quantifiers ranging over the standard elements of a standard set of any cardinality, hence they do not form an external set, but instead an external class denoted by $\mathcal{N}$. Similarly, the external numbers form an external class, denoted by $\mathbb{E}$.

The rules for addition, subtraction, multiplication and division of external numbers of Definition 2.3 below are in line with the rules of informal error analysis [20]. Here they are defined formally as Minkowski operations on sets of real numbers.

Definition 2.3. Let $a, b \in \mathbb{R}$, A, B be neutrices and $\alpha = a + A, \beta = b + B$ be external numbers.
(1) $\alpha \pm \beta = a \pm b + A + B$.
(2) $\alpha\beta = ab + Ab + Ba + AB$.
(3) If α is zeroless, $\frac{1}{\alpha} = \frac{1}{a} + \frac{A}{a^2}$.

If α or β are zeroless, in Definition 2.3(2) we may neglect the neutrix product AB. Definition 2.3(3) does not permit to divide by neutrices. However, division of neutrices is allowed in terms of division of groups.

Definition 2.4. Let $A, B \in \mathcal{N}$. Then we define

$$A : B = \{c \in \mathbb{R} | cB \subseteq A\}.$$

External neutrices are appropriate as a model for the Sorites property and orders of magnitude, for they are stable under some shifts, additions and multiplications. In particular, a neutrix N is invariant under multiplication by appreciable numbers, i.e., $@N = N$. An *absorber* of N is a real number a such that $aN \subset N$ and an *exploder* is a real number b such that $bN \supset N$. A zeroless external number α may be seen as a real number such that its error term $N(\alpha)$ is small; indeed its *relative imprecision* $R(\alpha) \equiv N(\alpha)/\alpha$ satisfies $R(\alpha) \subseteq \oslash$.

Notions such as limited, infinitesimal, absorber and exploder may be extended in a natural way to external numbers.

An order relation on the external numbers is given as follows.

Definition 2.5. Let $\alpha, \beta \in \mathbb{E}$. We define

$$\alpha \leq \beta \Leftrightarrow \forall a \in \alpha \exists b \in \beta (a \leq b).$$

If $\alpha \cap \beta = \emptyset$ and $\alpha \leq \beta$, then $\forall a \in \alpha \forall b \in \beta (a < b)$ and we write $\alpha < \beta$.

The relation $\leq$ is an order relation indeed, and compatible with the operations, with some small adaptations [18,16]. For neutrices A, B the order relation corresponds to inclusion. The inverse order relation is given by

$$\alpha \geq \beta \Leftrightarrow \forall a \in \alpha \exists b \in \beta (a \geq b),$$

and $\alpha > \beta$ if $\forall a \in \alpha \forall b \in \beta (a > b)$. Clearly $\alpha < \beta$ implies $\beta > \alpha$. However, both $\oslash \leq \pounds$ and $\oslash \geq \pounds$ hold. The notion of *maximum* will be defined for the relation $\leq$, i.e., if $\alpha \leq \beta$, then $\beta = \max(\alpha, \beta)$. If the neutrix A is contained in the neutrix B, one has $B = \max(A, B)$. External numbers α such that $0 \leq \alpha$ are called *non-negative*. The *absolute value* of an external number $\alpha = a + A$ is defined by $|\alpha| = |a| + A$.

By their close relation to the real numbers, practical calculations with external numbers tend to be quite straightforward. Nevertheless, some care is needed with distributivity; however, we always have subdistributivity. Necessary and sufficient conditions for distributivity are given in [21, Theorem 5.6].

Next proposition gives some useful properties of external numbers.

Proposition 2.6. [18,1] *Let $\alpha = a + A$ be a zeroless external number, γ be an external number, B be a neutrix and $n \in \mathbb{N}$ be standard. Then*

(1) $\alpha B = aB$ *and* $\frac{B}{\alpha} = \frac{B}{a}$.
(2) $N(1/\alpha) = N(\alpha)/\alpha^2$.
(3) $R(\alpha), R(1/\alpha) \subseteq \oslash$.
(4) $\alpha \cap \oslash\alpha = \varnothing$.
(5) $N(\alpha\gamma) = \alpha N(\gamma) + N(\alpha)\gamma$.
(6) $N((a + A)^n) = a^{n-1}A$.
(7) If *α is limited and is not an absorber of B, then* $\alpha B = \frac{B}{\alpha} = B$.

2.3 Gauss-Jordan operations

We will assume that the elementary matrices corresponding to the Gauss-Jordan operations have real coefficients. This reflects the common practice to use relatively simple numbers for these operations, and in this way more algebraic laws are respected. We give notations for the Gauss-Jordan operations and the intermediate matrices, and the explicit formulas in terms of quotients of minors.

We consider only square matrices and denote by $\mathcal{M}_n(\mathbb{R})$ the set of all $n \times n$ matrices over the field $\mathbb{R}$, with $n \in \mathbb{N}$, $n \geq 1$ being standard.

Definition 2.7. Let $\mathcal{A} = [a_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{R})$ be non-singular. For every q with $1 \leq q \leq 2n$, the *Gauss-Jordan operation matrix* $\mathcal{G}_q$ and the *intermediate matrix* $\mathcal{A}^{(q)}$ are defined as follows.

We let $\mathcal{G}_0$ be the $n \times n$ identity matrix I_n and $\mathcal{A}^{(0)} \equiv [a_{ij}^{(0)}]_{n\times n} = \mathcal{A}$. Let $q = 2k$ be even with $k < n$. We assume that $\mathcal{G}_{2k}$ and $\mathcal{A}^{(2k)} = [a_{ij}^{(2k)}]_{n\times n}$ are defined, where $a_{k+1k+1}^{(2k)} \neq 0$. We define for $1 \leq i, j \leq n$

$$g_{ij}^{(2k+1)} = \begin{cases} 1 & \text{if } i = j \neq k+1 \\ 0 & \text{if } i \neq j \\ \dfrac{1}{a_{k+1k+1}^{(2k)}} & \text{if } i = j = k+1 \end{cases}$$

The matrix $\mathcal{G}_{2k+1}$ is defined by

$$\mathcal{G}_{2k+1} = \left[g_{ij}^{(2k+1)} \right]_{n\times n},$$

and the matrix $\mathcal{A}^{(2k+1)}$ by

$$\mathcal{A}^{(2k+1)} = \mathcal{G}_{2k+1}\mathcal{A}^{(2k)} \equiv [a_{ij}^{(2k+1)}]_{n\times n}.$$

Then we define for $1 \leq i, j \leq n$

$$g_{ij}^{(2k+2)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } j \notin \{i, k+1\} \\ -a_{ik+1}^{(2k+1)} & \text{if } i \neq k+1, j = k+1, \end{cases}$$

the matrix $\mathcal{G}_{2k+2}$ by

$$\mathcal{G}_{2k+2} = \left[g_{ij}^{(2k+2)} \right]_{n\times n}$$

and the matrix $\mathcal{A}_{2k+2}$ by

$$\mathcal{A}^{(2k+2)} = \mathcal{G}_{2k+2}\mathcal{A}^{(2k+1)} \equiv [\alpha_{ij}^{(2k+2)}]_{n\times n}.$$

The matrix of odd order $\mathcal{G}_{2k+1}$ corresponds to the multiplication of row $k+1$ of $\mathcal{A}^{(2k)}$ by $1/a^{(2k)}_{k+1k+1}$, and the matrix of even order $\mathcal{G}_{2k+2}$ transforms the entries of column k of $\mathcal{A}^{(2k+1)}$ into zero, except for the entry $a^{(2k+1)}_{k+1k+1}(=1)$.

The condition that the pivots $a^{(2k)}_{k+1k+1}$ are non-zero can always be satisfied; in fact, they may always be chosen to be maximal, which is numerically desirable. In fact, the use of a pivot that is too small may lead to an outcome which exceeds the error bound of at least one inclusion of the system. This will be illustrated by Example 2.29 of Section 2.5.

A maximal pivot on the principal diagonal can always be obtained by interchanging rows and columns. This is a consequence of the next definition and propositions. We introduce first a notation for minors, taken from [22].

Notation 2.8. Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{R})$. For each $k \in \mathbb{N}$ such that $1 \le k \le n$, let $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_k \le n$.

(1) We denote the $k \times k$ submatrix of $\mathcal{A}$ consisting of the rows with indices $\{i_1, \dots, i_k\}$ and columns with indices $\{j_1, \dots, j_k\}$ by $\mathcal{A}^{i_1 \cdots i_k}_{j_1 \cdots j_k}$.

(2) We denote the corresponding $k \times k$ minor by

$$m^{i_1 \cdots i_k}_{j_1 \cdots j_k} = \det(\mathcal{A}^{i_1 \cdots i_k}_{j_1 \cdots j_k}).$$

(3) For $1 \le k \le n$ we may denote the principal minor of order k by m_k, i.e., $m_k = m^{1 \cdots k}_{1 \cdots k}$.

Definition 2.9. Assume $\mathcal{A} \in \mathcal{M}_n(\mathbb{R})$. Then $\mathcal{A}$ is called *properly arranged*, if $|a_{ij}| \le |a_{11}|$ for $1 \le i, j \le n$ and $|m^{1 \cdots ki}_{1 \cdots kj}| \le |m_{k+1}|$ for every k such that $1 \le k \le n-1$, whenever $k+1 \le i, j \le n$.

Proposition 2.10. *Let $n \ge 1$. Let $\mathcal{A} = [a_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{R})$. By interchanging rows and columns, if necessary $\mathcal{A}$ can be properly arranged.*

The proof of this proposition is straightforward.

Definition 2.11. Assume $\mathcal{A} = [a_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{R})$ is non-singular. Then $\mathcal{A}$ is called *diagonally eliminable* if $a^{(2k-2)}_{kk} \ne 0$ for $1 \le k \le n$.

Proposition 2.12. [9] *Let $n \ge 1$. Let $\mathcal{A} = [a_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{R})$ be non-singular and properly arranged. Then $\mathcal{A}$ is diagonally eliminable.*

It follows from Propositions 2.10 and 2.12 that if $\mathcal{A}$ is non-singular, we may assume without restriction of generality that it is diagonally eliminable.

Proofs of the following explicit expressions for the Gauss-Jordan operation matrices can be found in [8] and [9].

Theorem 2.13. (Explicit expressions for Gauss-Jordan operations). *Let $\mathcal{A} = [a_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{R})$ be diagonally eliminable. For $k < n$ the Gaussian elimination matrix of odd order $\mathcal{G}_{2k+1} = [g^{(2k+1)}_{ij}]_{n \times n}$ satisfies*

$$g^{(2k+1)}_{ij} = \begin{cases} 1 & \text{if } i = j \ne k+1 \\ 0 & \text{if } i \ne j \\ \dfrac{m_k}{m_{k+1}} & \text{if } i = j = k+1 \end{cases}$$

and the Gaussian elimination matrix of even order $\mathcal{G}_{2k+2} = \left[g^{(2k+2)}_{ij}\right]_{n \times n}$ satisfies

$$g_{ij}^{(2k+2)} = \begin{cases} 0 & \text{if } j \notin \{i, k+1\} \\ 1 & \text{if } j = i \\ (-1)^{k+i+1}\dfrac{m_{1\ldots i-1 i+1\ldots k+1}^{1\ldots k}}{m_k} & \text{if } 1 \le i \le k, j = k+1 \\ -\dfrac{m_{1\ldots kj}^{1\ldots ki}}{m_k} & \text{if } k+1 < i \le n, j = k+1. \end{cases}$$

Theorem 2.14. (Explicit expressions for intermediate matrices) *Let* $\mathcal{A} = [a_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{R})$ *be diagonally eliminable. Let* $k < n$. *Then*

$$\mathcal{A}^{(2k)} = \begin{bmatrix} 1 & \cdots & 0 & a_{1k+1}^{(2k)} & \cdots & a_{1n}^{(2k)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & a_{kk+1}^{(2k)} & \cdots & a_{kn}^{(2k)} \\ 0 & \cdots & 0 & a_{k+1k+1}^{(2k)} & \cdots & a_{k+1n}^{(2k)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nk+1}^{(2k)} & \cdots & a_{nn}^{(2k)} \end{bmatrix},$$

where

$$a_{ij}^{(2k)} = \begin{cases} (-1)^{k+i}\dfrac{m_{1\ldots i-1 i+1\ldots kj}^{1\ldots k}}{m_k} & \text{if } 1 \le i \le k, k+1 \le j \le n \\ \dfrac{m_{1\ldots kj}^{1\ldots ki}}{m_k} & \text{if } k+1 \le i \le n, k+1 \le j \le n. \end{cases}$$

In particular, $\mathcal{A}^{(2n)} = I_n$.

Finally, we consider the inverse Gauss-Jordan procedure. If $\mathcal{A}$ is diagonally eliminable, the inverses of the matrices of the Gauss-Jordan operations $\mathcal{G}_q$ are well-defined, as follows. For odd indices we have $\mathcal{G}_{2k+1}^{-1} = [(g_{ij}^{-1})^{(2k+1)}]_{n\times n}$, with

$$(g_{ij}^{-1})^{(2k+1)} = \begin{cases} 1 & \text{if } i = j \neq k+1 \\ 0 & \text{if } i \neq j \\ \dfrac{m_{k+1}}{m_k} & \text{if } i = j = k+1 \end{cases}$$

and for even indices $\mathcal{G}_{2k+2}^{-1} = \left[(g_{ij}^{-1})^{(2k+2)}\right]_{n\times n}$, with

$$(g_{ij}^{-1})^{(2k+2)} = \begin{cases} 0 & \text{if } j \notin \{i, k+1\} \\ 1 & \text{if } j = i \\ (-1)^{k+i}\dfrac{m_{1\ldots i-1 i+1\ldots k+1}^{1\ldots k}}{m_k} & \text{if } 1 \le i \le k, j = k+1 \\ \dfrac{m_{1\ldots kj}^{1\ldots ki}}{m_k} & \text{if } k+1 < i \le n, j = k+1. \end{cases}$$

2.4 Matrices with external numbers

We always suppose that $n \in \mathbb{N}$, $n \ge 1$ is standard. We denote by $\mathcal{M}_n(\mathbb{E})$ the class of all $n \times n$ matrices

$$\mathcal{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}, \tag{1}$$

where $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ for $1 \le i, j \le n$. The matrix $\mathcal{A}$ is called an *external matrix* and we use the usual notation $\mathcal{A} = [\alpha_{ij}]_{n\times n}$. A matrix $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ is said to be *neutricial* if all of its entries are neutrices, a special case is given by the zero matrix. With respect to (1) the matrix $P = [a_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{R})$ is called a *representative matrix* and the matrix $A = [A_{ij}]_{n\times n}$ the *associated neutricial matrix*. A matrix $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ with representative matrix equal to the identity matrix I_n and associated neutricial matrix contained in $[\oslash]_{n\times n}$ is called a *near-identity matrix*, and is denoted by $\mathcal{I}_n$; note that an identity matrix is a near-identity matrix.

For $\mathcal{A}, \mathcal{B} \in \mathcal{M}_n(\mathbb{E})$ we write $\mathcal{A} \subseteq \mathcal{B}$ if $\alpha_{ij} \subseteq \beta_{ij}$ for all i, j such that $1 \le i, j \le n$.

Notation 2.15. Let $\mathcal{A} = [\alpha_{ij}]_{n\times n} \equiv [a_{ij} + A_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{E})$. We define

$$|\overline{\alpha}| = \max_{1\le i,j\le n} |\alpha_{ij}|, \overline{A} = \max_{1\le i,j\le n} A_{ij}.$$

Definition 2.16. An external matrix $\mathcal{A}$ is said to be *limited* if $|\overline{\alpha}| \subset \pounds$ and *reduced* if $\overline{\alpha} = \alpha_{11}$ and $\alpha_{11} = 1 + A_{11}$, with $A_{11} \subseteq \oslash$, while all other entries have representatives which in absolute value are at most 1.

By the second part of Definition 2.16 a reduced external matrix always has a reduced representative matrix.

For $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$, the determinant $\Delta \equiv \det(\mathcal{A}) \equiv d + D$ is defined in the usual way through sums of signed products [23].

Proposition 2.17. *Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ be limited. Then $\Delta \subset \pounds$.*

Proof. One has $|\Delta| \le n!|\overline{\alpha}|^n \subset n!\pounds^n = \pounds$. □

Definition 2.18. Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$. Then $\mathcal{A}$ is called *non-singular* if Δ is zeroless.

Observe that a representative matrix of a non-singular matrix $\mathcal{A}$ is always non-singular. It is not true in general that $\det(\mathcal{A})$ is equal to the set of determinants of representative matrices. This is shown in [17], which contains an overview of the calculus of matrices with external numbers and thier determinants. We will use the following property of inclusion, which is not contained in [17].

Proposition 2.19. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{M}_n(\mathbb{E})$. If $\mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{A}\mathcal{B} \subseteq \mathcal{A}\mathcal{C}$.*

Proposition 2.19 is an immediate consequence of the fact that, given external numbers α, β, γ such that $\alpha \subseteq \beta$, one has $\gamma\alpha \subseteq \gamma\beta$.

Let $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be an external matrix. The Gauss-Jordan operations will always be carried out using the elements of a representative matrix $P = [a_{ij}]_{n\times n}$. In particular, the notions of properly arranged and diagonally eliminable are defined by reference to representative matrices.

Definition 2.20. Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ be reduced and non-singular.

(1) We say that $\mathcal{A}$ is *properly arranged* if it has a properly arranged representative matrix P. In this case, we say that $\mathcal{A}$ is properly arranged with respect to P.

(2) If $\mathcal{A}$ has a diagonally eliminable representative matrix P, we say that $\mathcal{A}$ is *diagonally eliminable with respect to P*.

Because $\mathcal{A}$ has a reduced representative matrix, by Proposition 2.10 we may assume without restriction of generality that a properly arranged representative matrix P is reduced. The matrix P is non-singular, for it is a representative matrix of the non-singular matrix $\mathcal{A}$. Hence, P is diagonally eliminable by Proposition 2.12.

2.5 Flexible systems and stability

We define flexible systems of linear equations in terms of inclusions, where we restrict ourselves to the case where the number of equations is equal to the number of variables. For the particular case of non-singular systems we define a notion of stability. We state the main tool for the proof of Theorem 1.1, which says that if the coefficient matrix is properly arranged and the neutrices at the right-hand side are all equal, a Gauss-Jordan operation transforms a stable system into a stable system, until obtaining a system with a coefficient matrix in the form of a near-identity matrix and a right-hand side with still the same neutrices.

Definition 2.21. Let $n \in \mathbb{N}$ be standard and $\xi_1, \dots, \xi_n$ be external numbers. Then $\xi \equiv (\xi_1, \dots, \xi_n)^T$ is called an *external vector*. For $1 \le i \le n$, let $\xi_i = x_i + X_i$. Then $x \equiv (x_1, \dots, x_n)^T$ is called a *representative vector* and $X \equiv (X_1, \dots, X_n)^T$ is called the *associated neutricial vector*, so $\xi = x + X$. We write $|\overline{\xi}| = \max(|\xi_1|, \dots, |\xi_n|)$.

Definition 2.22. A (*square*) *flexible system* is a system of inclusions

$$\begin{cases} \alpha_{11}x_1 + & \alpha_{12}x_2 + & \cdots & +\alpha_{1n}x_n & \subseteq \beta_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n1}x_1 + & \alpha_{n2}x_2 + & \cdots & +\alpha_{nn}x_n & \subseteq \beta_n, \end{cases} \tag{2}$$

where n is a standard natural number, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\alpha_{ij} \equiv a_{ij} + A_{ij}$, $\beta_i \equiv b_i + B_i$ are external numbers for $1 \le i, j \le n$. We denote the matrix $[\alpha_{ij}]_{n\times n}$ by $\mathcal{A}$, the representative matrix $[a_{ij}]_{n\times n}$ by P, the associated neutricial matrix $[A_{ij}]_{n\times n}$ by A, the external vector $(\beta_i, \dots, \beta_n)^T$ by $\mathcal{B}$ and the system (2) by $\mathcal{A}|\mathcal{B}$. The set

$$S \equiv \{x \in \mathbb{R}^n | \mathcal{A}x \subseteq \mathcal{B}\}$$

is called the *solution* of the inclusion $\mathcal{A}|\mathcal{B}$. The solution is *exact* if $\mathcal{A}S = \mathcal{B}$.

Remark 2.23. In [1] and [23], flexible systems have been equivalently defined by inclusions of the form $\mathcal{A}\xi \subseteq \mathcal{B}$, where ξ is an external vector. Now $\mathcal{A}\xi \subseteq \mathcal{B}$ if and only if $x \in S$ whenever $x \in \xi$. Indeed, clearly $\mathcal{A}\xi \subseteq \mathcal{B}$ implies that $\mathcal{A}x \subseteq \mathcal{B}$ for all $x \in \xi$, hence $x \in S$ for all $x \in \xi$. Conversely, let $\xi = (\xi_1, \dots, \xi_n)$. Let $1 \le i \le n$ and assume that $\alpha_{i1}x_1 + \dots + \alpha_{in}x_n \subseteq \beta_i$ whenever $x_1 \in \xi_1, \dots, x_n \in \xi_n$. Let $t \in \tau \equiv \alpha_{i1}\xi_1 + \dots + \alpha_{in}\xi_n$. It follows from the definition of the Minkowski operations that for all j with $1 \le j \le n$ there exist $a_{ij} \in \alpha_{ij}$ and $x_j \in \xi_j$ such that $t = a_{i1}x_1 + \dots + a_{in}x_n$. Then $t \in \beta_i$. So $\tau \subseteq \beta_i$, hence $\mathcal{A}\xi \subseteq \mathcal{B}$. As a consequence, we may restrict ourselves to the systems (2), i.e., flexible systems with real variables.

We now introduce some terminology, in particular we extend some of the notions on matrices of Section 2.4 to systems of equations.

Definition 2.24. The system $\mathcal{A}|\mathcal{B}$ is called *reduced* if $\mathcal{A}$ is reduced, *homogeneous* if $\mathcal{B}$ is a neutrix vector, *upper homogeneous* if $|\overline{\beta}|$ is a neutrix and *uniform* if the neutrices of the right-hand side $B_i \equiv B$ are all the same. The system is called *non-singular* if $\mathcal{A}$ is non-singular, *properly arranged*, respectively, *diagonally eliminable* (with respect to a matrix of representatives P) if $\mathcal{A}$ is properly arranged, respectively, diagonally eliminable with respect to P.

The following examples show that some caution is needed when solving flexible systems, for there may be a mismatch between the neutrices at the left-hand side and the right-hand side. Indeed, when substituting a solution-candidate into the system, it does not need to be exact, i.e., some neutrix at the left-hand side may be too small. It is also possible that a solution-candidate is not feasible, i.e., the size of its neutrices is such that, when substituting it into the system at least one of the inclusions is not satisfied. This may also happen when applying Gauss-Jordan operations to a flexible system, even in the non-singular case.

Example 2.25. The inclusion $\oslash x \subseteq \pounds$ does not have an exact solution. Indeed, it is satisfied by all limited numbers, but not by any unlimited number. Hence, $S = \pounds$, with $\oslash \pounds = \oslash \subset \pounds$.

Example 2.26. Let $\varepsilon \simeq 0$, $\varepsilon \neq 0$. The inclusion $(1 + \oslash)x \subseteq 1 + \varepsilon\pounds$ has no solution.

Example 2.27. Consider the flexible system

$$\begin{cases} (1 + \oslash)x + (1 + \varepsilon\oslash)y \subseteq \oslash \\ (1 + \varepsilon\pounds)x - (1 + \varepsilon\pounds)y \subseteq \varepsilon\pounds \end{cases} \tag{3}$$

To show that the Gauss-Jordan operations may not respect feasibility, we subtract the first inclusion from the second. Then we obtain

$$\begin{cases} (1 + \oslash)x + (1 + \varepsilon\oslash)y \subseteq \oslash \\ \oslash x - 2(1 + \varepsilon\pounds)y \subseteq \oslash. \end{cases}$$

The obvious solution is the external (neutricial) vector $K \equiv (\oslash, \oslash)$, but due to the fact that one of the neutrices at the right-hand side has been increased, it does no longer satisfy the original system. For instance, the representative vector $(0, \sqrt{\varepsilon})$ does not satisfy the second inclusion of (3).

As shown in [23] the solution is given by

$$N = \oslash \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \varepsilon\pounds \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix},$$

which is strictly contained in K. Observe that N is not an neutricial vector, though it is the result of applying a rotation to the neutricial vector $(\oslash, \varepsilon\pounds)$.

Example 2.28. The Gauss-Jordan procedure may also lead to non-feasibility, if the determinant of the coefficient matrix is too small. Indeed, consider the system

$$\begin{cases} x_1 + x_2 \subseteq 1 + \oslash \\ \varepsilon x_2 \subseteq \oslash, \end{cases}$$

with $\varepsilon \simeq 0$, $\varepsilon \neq 0$. The determinant of the coefficient matrix $\Delta = \varepsilon$ is an absorber of the neutrix of the right-hand side $B = \oslash$. Applying Gauss-Jordan elimination we blow B up to $\oslash/\varepsilon$, and obtain at the right-hand side the neutrix vector $(\oslash/\varepsilon, \oslash/\varepsilon)^T$, which is obviously not feasible.

Example 2.29. Also the use of non-maximal pivots in Gaussian elimination may lead to non-feasibility. Again, let $\varepsilon > 0$ be an infinitesimal and consider the system

$$\begin{cases} \varepsilon x + y \subseteq \oslash \\ x + y \subseteq \oslash. \end{cases} \tag{4}$$

Let $P \equiv [a_{ij}]_{2\times 2} = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix}$ be the coefficient matrix. Applying Gaussian elimination we first divide both sides of the first row by the pivot $a_{11} = \varepsilon$, then the system becomes

$$\left[\begin{array}{cc|c} 1 & 1/\varepsilon & \oslash/\varepsilon \\ 1 & 1 & \oslash \end{array}\right].$$

Subtracting the first row from the second row, we obtain

$$\left[\begin{array}{cc|c} 1 & 1/\varepsilon & \oslash/\varepsilon \\ 0 & 1 - 1/\varepsilon & \oslash/\varepsilon \end{array}\right].$$

After dividing the second row by $1 - 1/\varepsilon$ the system becomes

$$\left[\begin{array}{cc|c} 1 & 1/\varepsilon & \oslash/\varepsilon \\ 0 & 1 & \oslash \end{array}\right],$$

which has the solution $(x, y)^T = (\oslash/\varepsilon, \oslash)^T$; however, this does not satisfy the second inclusion of (4).

On the other hand, by interchanging the rows the system becomes

$$\left[\begin{array}{cc|c} 1 & 1 & \oslash \\ \varepsilon & 1 & \oslash \end{array}\right].$$

Applying Gaussian elimination to the first column, using the pivot 1 which is now maximal, we obtain $\left[\begin{array}{cc|c} 1 & 1 & \oslash \\ 0 & 1-\varepsilon & \oslash \end{array}\right]$. Then dividing the second row by $1 - \varepsilon \simeq 1$ leads to the system $\left[\begin{array}{cc|c} 1 & 1 & \oslash \\ 0 & 1 & \oslash \end{array}\right]$ with correct solution $(x, y)^T = (\oslash, \oslash)^T$.

We will define stability conditions, which ensure that the Gauss-Jordan operations respect feasibility. We recall first the following notions of relative imprecision of external vectors and matrices from [1].

Definition 2.30. Let $\mathcal{B} = (\beta_1, \ldots, \beta_n)^T$ be an external vector. Let $\underline{B} = \min_{1\le i\le n} B_i$. If $|\overline{\beta}|$ is zeroless, its *relative imprecision* $R(\mathcal{B})$ is defined by

$$R(\mathcal{B}) = \underline{B}/|\overline{\beta}|.$$

In the special case that $\overline{\beta} = B$ for some neutrix B, we define

$$R(\mathcal{B}) = \underline{B} : B.$$

Observe that, whenever $1 \le i \le n$, it holds that

$$R(\mathcal{B})\beta_i \subseteq \underline{B}, \tag{5}$$

and that $R(\mathcal{B}) \subseteq \oslash$ if the system is not upper homogeneous.

Definition 2.31. Let $\mathcal{A} = [\alpha_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{E})$ be a limited, non-singular matrix. Its *relative imprecision* $R(\mathcal{A})$ is defined by $R(\mathcal{A}) = \overline{A}/\Delta$. The matrix $\mathcal{A}$ is called *stable* if $R(\mathcal{A}) \subseteq \oslash$.

The biggest neutrix $\overline{A}$ occurring in a limited matrix $\mathcal{A}$ is always contained in $\oslash$, but if Δ is infinitesimal, for the matrix to be stable, the entries need to be sharper.

Definition 2.32. Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ be limited and non-singular. The system $\mathcal{A}|\mathcal{B}$ is said to be *stable* if
(1) $\mathcal{A}$ is stable.
(2) $R(\mathcal{A}) \subseteq R(\mathcal{B})$.
(3) Δ is not an absorber of $\underline{B}$.

Condition (2) expresses that in a sense the coefficient matrix is more precise than the right-hand side. Conditions (1) and (3) express that the determinant Δ, which of course must be non-zero, should not be too small. Indeed, as we will see in Section 3.1, when applying Gauss-Jordan operations Δ appears in the denominator. Now dividing by Δ should not explode the smallest neutrix at the right-hand side, and lead only to a moderate increase of the neutrices in the coefficient matrix, which should remain small with respect to the determinant.

If Condition (1) is not satisfied, we even have $R(\mathcal{B}) \supseteq R(\mathcal{A}) \supseteq \pounds$, which means that the system must be upper homogeneous. This is very restrictive, for instance uniform systems will be necessarily homogeneous. With respect to Condition (2), Example 2.26 shows that a flexible system such that the relative imprecision of the coefficient matrix contains strictly the relative imprecision of the right-hand side may have no solution at all.

It will be seen that the notion of stability is respected by the steps of the Gauss-Jordan procedure, resulting in the same imprecision for the solution and the right-hand side.

We introduce now some notations with respect to the Gauss-Jordan procedure for flexible systems.

Notation 2.33. Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ be diagonally eliminable with respect to a representative matrix P. For $1 \le q \le 2n$ we denote the qth Gauss-Jordan operation matrix with respect to P by $\mathcal{G}_q^P$ and we write $\mathcal{G}^P\mathcal{A} = \mathcal{G}_{2n}^P(\mathcal{G}_{2n-1}^P \cdots (\mathcal{G}_2^P(\mathcal{G}_1^P\mathcal{A})))$.

We will see that under the conditions of stability of Definition 2.31 the result of the Gauss-Jordan procedure does not depend on the choice of the representative matrix, and we may simply write $\mathcal{G}^P\mathcal{A} = \mathcal{G}\mathcal{A}$. We may also write $\mathcal{G}_q^P = \mathcal{G}_q$ for $1 \le q \le 2n$ if the matrix P is made clear by the context. Then we also adopt the notation of Definition 2.7 for the intermediate matrices, i.e., we have

$$\begin{aligned}\mathcal{G}_q^P(\mathcal{G}_{q-1}^P \cdots (\mathcal{G}_1^P\mathcal{A})) &= \mathcal{G}_q(\mathcal{G}_{q-1} \cdots (\mathcal{G}_1\mathcal{A})) \equiv \mathcal{A}^{(q)} \\ &\equiv [\alpha_{ij}^{(q)}]_{n\times n} \equiv [a_{ij}^{(q)} + A_{ij}^{(q)}]_{n\times n} = P^{(q)} + [A_{ij}^{(q)}]_{n\times n}.\end{aligned}$$

Notation 2.34. Suppose that $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ and that the system $\mathcal{A}|\mathcal{B}$ is non-singular, properly arranged with respect to a matrix of representatives P and uniform. We write $\mathcal{B}^{(0)} = [\beta_1^{(0)}, \cdots, \beta_n^{(0)}]^T = \mathcal{B}, [B] = [B, B, \ldots, B]^T$ and $[B]^{(0)} = [B^{-1}]^{(0)} = [B]$. For $1 \le q \le 2n$, we write

$$\begin{aligned}\mathcal{B}^{(q)} &= \mathcal{G}_q^P(\mathcal{G}_{q-1}^P \cdots (\mathcal{G}_1^P\mathcal{B})) \equiv [\beta_1^{(q)}, \ldots, \beta_n^{(q)}]^T, \\ \mathcal{G}^P\mathcal{B} &= \mathcal{G}_{2n}^P(\mathcal{G}_{2n-1}^P \cdots (\mathcal{G}_1^P\mathcal{B})), \\ |\overline{\beta^{(q)}}| &= \max_{1\le i\le n}|\beta_i^{(q)}|, \\ [B]^{(q)} &= \mathcal{G}_q^P(\mathcal{G}_{q-1}^P \cdots (\mathcal{G}_1^P[B])).\end{aligned}$$

We see that after q Gauss-Jordan operations the system becomes $\mathcal{A}^{(q)}|\mathcal{B}^{(q)}$, and for $q = 2n$ the Gauss-Jordan procedure ends with $\mathcal{G}^P\mathcal{A}|\mathcal{G}^P\mathcal{B}$.

Theorem 2.35. *Suppose that the flexible system $\mathcal{A}|\mathcal{B}$ is properly arranged with respect to a representative matrix P and stable. Then*

(1) *The intermediate system $\mathcal{A}^{(q)}|\mathcal{B}^{(q)}$ is stable for all q such that $0 \le q \le 2n$.*
(2) *In particular, $\mathcal{G}^P\mathcal{A}|\mathcal{G}^P\mathcal{B}$ is stable, and $\mathcal{G}^P\mathcal{A}$ is a near-identity matrix.*

Theorem 2.35 will be shown in Section 3.1.

2.6 Solution sets and the Main theorem

The solution of a classical system of linear equations is the same as the solution obtained by the Gauss-Jordan procedure and, in the case of non-singular systems, equal to the solution given by Cramer's rule. This is not always true for flexible systems. However, the Main theorem (Theorem 1.1) states that under the conditions of stability the solutions of flexible systems still are all equal.

Definition 2.36. Assume the system $\mathcal{A}|\mathcal{B}$ is properly arranged with respect to a representative matrix P of $\mathcal{A}$. The *Gauss-Jordan solution* G^P of $\mathcal{A}|\mathcal{B}$ with respect to P is defined by

$$G^P = \{x \in \mathbb{R}^n | (\mathcal{G}^P\mathcal{A})x \subseteq \mathcal{G}^P\mathcal{B}\}.$$

If G^P does not depend on the choice of P, we simply call it the Gauss-Jordan solution, denoted by G.

Convention 2.37. From now on we always suppose that the system $\mathcal{A}|\mathcal{B}$ is non-singular, reduced and uniform.

As for non-singular systems, only the condition of uniformity is restrictive. In the context of the Gauss-Jordan procedure the condition is essential, since the simple addition of equations may have the effect that the solution of the resulting system is no longer feasible for the original system, see Example 2.27. By transforming a system with different neutrices $B_1, \ldots, B_n$ in the right-hand side into the system with neutrices at the right-hand side equal to $\underline{B}$, we obtain a uniform system, whose solutions are always feasible with respect to the original system.

Theorem 2.38 relates S with the result of the Gauss-Jordan procedure applied to $\mathcal{A}|\mathcal{B}$. Let P be a representative matrix of $\mathcal{A}$. Theorem 2.38(1) says that $S \subseteq G^P$. Example 2.28 shows that the converse is not always true, because the neutrices of G^P may be bigger, in particular if $\det(\mathcal{A})$ is small. Theorem 2.38(2) states that equality holds if $\det(\mathcal{A})$ is not an absorber of the neutrix B at the right-hand side and gives an effective way to find the solution. Like in the real case, the solution of $\mathcal{A}|\mathcal{B}$ may be determined by the Gauss-Jordan procedure. Moreover, the Gauss-Jordan solution does not depend on the choice of the representative matrix P of $\mathcal{A}$, provided it is reduced and properly arranged.

Theorem 2.38. *Assume the system $\mathcal{A}|\mathcal{B}$ is properly arranged with respect to a representative matrix P of $\mathcal{A}$. Then*
(1) $S \subseteq G^P$.
(2) *If Δ is not an absorber of B , the Gauss-Jordan solution G is well-defined and $G^P = G = S$.*

Theorem 2.38 will be proved in Section 3.3.

We consider now the relation between the forms of solution mentioned above and Cramer's rule.

Definition 2.39. Consider the system $\mathcal{A}|\mathcal{B}$. Let M_i be the matrix obtained from $\mathcal{A}$ by the substitution of the ith column by $\mathcal{B}$. Then the external vector

$$y^T = \left(\frac{\det(M_1)}{\Delta}, \ldots, \frac{\det(M_n)}{\Delta}\right)^T \tag{6}$$

is called the *Cramer solution*.

It was shown in [23] that the set of determinants of representative matrices of an external matrix $\mathcal{M}$ is included in $\det(\mathcal{M})$, and an example is given where the inclusion is strict. Then it follows from the fact that Cramer's rule solves ordinary linear systems that $S \subseteq y$, and in some cases $S \subset y$.

Theorem 2.40 gives conditions for Cramer's rule to be valid for flexible systems. In [1], the theorem was shown for non-homogeneous flexible systems, so Theorem 2.40 implies that Cramer's rule is also valid for homogeneous systems. It will be proved in Section 3.2.

Theorem 2.40. (Cramer's rule for flexible systems) *Consider the system $\mathcal{A}|\mathcal{B}$. If*
(1) $R(\mathcal{A}) \subseteq R(\mathcal{B})$
(2) Δ *is not an absorber of B,*

its solution is given by the external vector (6).

The Cramer solution given by (6) is an explicit formula for the solution of a flexible system $\mathcal{A}|\mathcal{B}$ and, in case of stability, by the Main theorem 1.1 it is equal to the Gauss-Jordan solution G. The latter solution results from a well-defined stepwise procedure, once $\mathcal{A}$ is properly arranged with respect to a reduced representative matrix P, which we may choose arbitrarily. The Gauss-Jordan solution can even be made more explicit, for it is equal to $\mathcal{G}^P\mathcal{B}$, i.e., the result of the Gauss-Jordan procedure applied to the right-hand side $\mathcal{B}$. The Main theorem will be proved in Section 3.4.

3 Proofs

This section is dedicated successively to the proofs of Theorems 2.35, 2.40 and 2.38, finally leading to the proof of Main theorem.

3.1 Proof of Theorem 2.35

We will see that if the coefficient matrix is reduced, properly arranged and stable, the application of the Gauss-Jordan operations to the intermediate matrices leads to at most a moderate growth for the elements and their neutrix parts, implying that these matrices remain stable. If the determinant is not an absorber of the neutrix part of the right-hand side, the latter neutrix even remains constant. For stable systems, the relative imprecision of the intermediate matrices remains always less than the relative imprecision of the right-hand side. In order to establish the above properties of orders of magnitude and stability we derive bounds on the size of determinants and minors. Indeed, because the pivots are quotients of minors, they have direct influence on the order of magnitude of the entries and neutrix parts of the intermediate matrices and the right-hand side. Together this leads to a proof of Theorem 2.35 on the preservation of stability under the Gauss-Jordan operations, with the final matrix being a near-identity matrix.

We study first how the Gauss-Jordan procedure acts on the coefficient matrix of the system $\mathcal{A}|\mathcal{B}$.

Remark 3.1. We recall from the previous section that a reduced matrix $\mathcal{A}$ has always a reduced representative matrix, and from now on we always suppose that a representative matrix is reduced.

Proposition 3.2 shows that the Gauss-Jordan operations do not lead to an unlimited growth of the elements of the intermediate matrices.

Proposition 3.2. *Let $\mathcal{A} = [\alpha_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{E})$ be a reduced, non-singular matrix that admits a properly arranged representative matrix $P = [a_{ij}]_{n\times n}$. Then $a_{ij}^{(q)}$ is limited whenever $1 \le q \le 2n$ and $1 \le i, j \le n$.*

Proof. The proof is by external induction. Because P is reduced, it holds that $|a_{ij}| \le 1$ for $1 \le i, j \le n$ and, since $a_{ij}^{(1)} = a_{ij}$, the same is true for $|a_{ij}^{(1)}|$. It follows that $|a_{1j}^{(2)}| = |a_{1j}| \le 1$ for $1 \le j \le n$ and $|a_{ij}^{(2)}| = |a_{ij} - a_{i1}a_{1j}| \le |a_{ij}| + |a_{i1}||a_{1j}| \le 2$ for $2 \le i \le n$, $1 \le j \le n$. Hence, $a_{ij}^{(2)}$ is limited for $1 \le i, j \le n$. As for the induction step, let $k \le n-1$ and suppose that $a_{ij}^{(q)}$ is limited for $q \le 2k$ and $1 \le i, j \le n$. Because the jth column of $P^{(2k+1)}$ is a unit vector for $1 \le j \le k$, the entries of these columns are limited. For $1 \le i \le n$, $k+1 \le j \le n$ one has

$$a_{ij}^{(2k+1)} = \begin{cases} a_{ij}^{(2k)} & \text{if } i \neq k+1 \\ \dfrac{m_{1\ldots kj}^{1\ldots ki}}{m_{k+1}} & \text{if } i = k+1. \end{cases}$$

So $a_{ij}^{(2k+1)} = a_{ij}^{(2k)}$ is limited for $i \neq k+1$ and $k+1 \le j \le n$ by the induction hypothesis, and because P is properly arranged, also for $i = k+1$ and $k+1 \le j \le n$, since $|a_{k+1j}^{(2k+1)}| \le \left|\frac{m_{1\ldots kj}^{1\ldots ki}}{m_{k+1}}\right| \le 1$. Combining, we see that $a_{ij}^{(2k+1)}$ is limited for $1 \le i, j \le n$.

As for $P^{(2k+2)}$, in addition to the first k columns, also the $(k+1)$th column is a unit vector, i.e., has limited components. Because the elements of $P^{(2k+1)}$ are limited we derive that $a_{k+1j}^{(2k+2)} = a_{k+1j}^{(2k+1)}$ is limited for $k+2 \le j \le n$, and $a_{ij}^{(2k+2)} = a_{ij}^{(2k+1)} - a_{ik+1}^{(2k+1)}a_{k+1j}^{(2k+1)}$ is limited for $1 \le i \le n$, $i \neq k+1$ and $k+1 \le j \le n$. Hence, $a_{ij}^{(2k+2)}$ is limited for $1 \le i, j \le n$. ☐

Theorem 3.5 gives bounds for the quotients $\frac{m_{k+1}}{m_k}$ of two successive principal minors of a representative matrix of the coefficient matrix. They are at least of the same order of magnitude as the determinant of the coefficient matrix and at most limited. These bounds, together with bounds for their multiplicative inverses, permit to derive Theorem 3.6 which gives bounds for the pivots and entries of the elementary matrices of the Gauss-Jordan procedure and the inverse procedure; we will need the latter to verify that the Gauss-Jordan solution is a solution of the original system. To prove Theorem 3.5 we present first some notation and an auxiliary result, saying that the determinants of the intermediate matrices are at least of the same order of magnitude as the determinant of the original matrix.

Notation 3.3. Let $\mathcal{A} = [\alpha_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{E})$ be a reduced, non-singular matrix that admits a properly arranged representative matrix $P = [a_{ij}]_{n\times n}$. For $1 \le q \le 2n$ we write $d = \det(P)$, $d^{(q)} = \det(P^{(q)})$ and $\Delta^{(q)} = \det \mathcal{A}^{(q)} = d^{(q)} + D^{(q)}$. For $1 \le k \le n$ the principal minor of order k of P is denoted by m_k.

Lemma 3.4. *Let $\mathcal{A} = [\alpha_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{E})$ be a reduced, non-singular matrix that admits a properly arranged representative matrix P. Let $1 \le q \le 2n$ and k be such that $q = 2k-1$ or $q = 2k$. Then $|d^{(q)}| = \left|\frac{d}{m_k}\right| > \oslash\Delta$.*

Proof. Let $1 \le k \le n$ and $q = 2k-1$ or $q = 2k$. In both cases

$$|d^{(q)}| = |\det(\mathcal{G}_q)\det(\mathcal{G}_{q-1})\cdots\det(\mathcal{G}_1)d| = \left|\frac{m_{k-1}}{m_k}\frac{m_{k-2}}{m_{k-1}}\cdots\frac{m_1}{m_2}\frac{1}{m_1}d\right| = \left|\frac{d}{m_k}\right|. \tag{7}$$

Suppose $|d^{(q)}| \subseteq \oslash\Delta$. Then $d \in m_k\oslash\Delta$ by (7). By Proposition 2.17 it holds that $d \in \oslash\Delta$. Hence, $d \in \oslash\Delta \cap \Delta$. Because Δ is zeroless, this contradicts Proposition 2.6(4). Hence, $|d^{(q)}| > \oslash\Delta$. □

Theorem 3.5. *Let $\mathcal{A} = [\alpha_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{E})$ be a reduced, non-singular matrix, which is properly arranged with respect to a matrix of representatives P. Then for $1 \le k < n$*

$$\oslash\Delta < \left|\frac{m_{k+1}}{m_k}\right| \in \pounds \tag{8}$$

and

$$\oslash < \left|\frac{m_k}{m_{k+1}}\right| \in \frac{\pounds}{\Delta}. \tag{9}$$

Proof. For $1 \le k \le n-1$ we have

$$\mathcal{A}^{(2k)} = \begin{bmatrix} 1+A_{11} & A_{12}^{(2k)} & \cdots & A_{1k}^{(2k)} & \alpha_{1(k+1)}^{(2k)} & \cdots & \alpha_{1n}^{(2k)} \\ A_{21}^{(2k)} & 1+A_{22}^{(2k)} & \cdots & \alpha_{2k}^{(2k)} & \alpha_{2(k+1)}^{(2k)} & \cdots & \alpha_{2n}^{(2k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{k1}^{(2k)} & A_{k2}^{(2k)} & \cdots & 1+A_{kk}^{(2k)} & \alpha_{k(k+1)}^{(2k)} & \cdots & \alpha_{kn}^{(2k)} \\ A_{(k+1)1}^{(2k)} & A_{(k+1)2}^{(2k)} & \cdots & A_{(k+1)k}^{(2k)} & \alpha_{(k+1)(k+1)}^{(2k)} & \cdots & \alpha_{(k+1)n}^{(2k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{n1}^{(2k)} & A_{n2}^{(2k)} & \cdots & A_{nk}^{(2k)} & \alpha_{n(k+1)}^{(2k)} & \cdots & \alpha_{nn}^{(2k)} \end{bmatrix}.$$

To prove (8), we suppose on the contrary that $a_{k+1k+1}^{(2k)} = \frac{m_{k+1}}{m_k} \in \oslash\Delta$. From $|a_{ij}^{(2k)}| = \left|\frac{m^{1\ldots ki}_{1\ldots kj}}{m_k}\right| \le \left|\frac{m_{k+1}}{m_k}\right| = |a_{k+1k+1}^{(2k)}|$ for $k+1 \le i, j \le n$ one derives that $a_{ij}^{(2k)} \in \oslash\Delta$ for $k+1 \le i,\ j \le n$. Let S_{n-k} be the set of all permutations of $\{k+1,\ldots,n\}$. Because Δ is limited,

$$d^{(2k)} = \sum_{\sigma\in S_{n-k}} \operatorname{sgn}(\sigma)a_{k+1\sigma(k+1)}^{(2k)}\cdots a_{n\sigma(n)}^{(2k)} \in (\oslash\Delta)^{n-k} \subseteq \oslash\Delta,$$

in contradiction to Lemma 3.4. Hence, $\left|\frac{m_{k+1}}{m_k}\right| > \oslash\Delta$.

Also, by Theorem 2.13 we have $\frac{m_{k+1}}{m_k} = a^{(2k+2)}_{k+1k+1}$, and the latter is limited by Proposition 3.2. Hence, formula (8) holds.

To complete the proof, it follows from (8) that $\oslash\Delta < \left|\frac{m_{k+1}}{m_k}\right| < \not{o}$. Hence, $\oslash < \left|\frac{m_k}{m_{k+1}}\right| < \frac{\not{o}}{\Delta}$. This implies (9). □

Theorem 3.6. *Let $\mathcal{A} = [\alpha_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{E})$ be a reduced, non-singular matrix, which is properly arranged with respect to a matrix of representatives P.*

(1) *Let $1 \le k < n$. Then the kth diagonal element of $\mathcal{G}^P_{2k+1}$ satisfies $g^{(2k+1)}_{k+1k+1} \in \frac{\pounds}{\Delta}$ and the elements of $\mathcal{G}^P_{2k+2}$ are all limited.*

(2) *All elements of the matrices $(\mathcal{G}^{-1})^P_q$, $1 \le q \le 2n$ of the inverse Gauss-Jordan procedure are limited.*

Proof. (1). The property is a direct consequence of Theorem 3.5 and Proposition 3.2.

(2). For the intermediate matrices of odd index the property follows from (8), and for the intermediate matrices of even index $q = 2k$, $k < n$ the property follows from the fact that $|(g^{-1})^{(2k+2)}_{ik+1}| = |g^{(2k+2)}_{ik+1}|$ for $1 \le i \le n$, $1 \le k \le n-1$, and Part 1. □

Proposition 3.7 gives estimates for the principal minors of the coefficient matrix and Proposition 3.8 gives a bound for the possible increase of the neutrix parts of the intermediate matrices of the Gauss-Jordan procedure.

Proposition 3.7. *Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ be a reduced, non-singular matrix, which is properly arranged with respect to a matrix of representatives P. Then for all k such that $1 \le k \le n$,*

$$\oslash\Delta^k < m_k \in \pounds.$$

Proof. Because P is reduced and properly arranged, it holds that $m_1 = a_{11} = 1 \in \pounds$. By Proposition 2.17 one has $\Delta \subset \pounds$. It follows that $\oslash\Delta < 1 = m_1 \subset \pounds$. Then for $1 < k \le n$ the identity

$$m_k = \frac{m_k}{m_{k-1}}\frac{m_{k-1}}{m_{k-2}}\cdots\frac{m_2}{m_1}m_1$$

and Theorem 3.5 imply that

$$\oslash\Delta^k < m_k \in \pounds^k = \pounds.$$ □

Proposition 3.8. *Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ be a reduced, non-singular matrix, which is properly arranged with respect to a matrix of representatives P. Then for all k such that $1 \le k \le n$,*

$$\overline{A^{(2k)}} = \overline{A^{(2k-1)}} \subseteq \frac{\overline{A}}{m_k} \subseteq \frac{\overline{A}}{\Delta^k}.$$

Proof. The proof is by external induction. For $k = 1$, because $m_1 = a_{11} = 1$, one has $\overline{A^{(2k-1)}} = \overline{A^{(1)}} = \overline{A} = \frac{\overline{A}}{m_1}$. By Part 1 of Theorem 3.6 it holds that $g^{(2k)}_{ij} = g^{(2)}_{ij}$ is limited for $1 \le i, j \le n$, hence $\overline{A^{(2)}} = \overline{A^{(1)}} = \frac{\overline{A}}{m_1}$.

As for the induction step, let $k < n$ and suppose that $\overline{A^{(2k-1)}} = \overline{A^{(2k)}} \subseteq \frac{\overline{A}}{m_k}$. Then $\overline{A^{(2k+1)}} \subseteq \frac{m_k}{m_{k+1}}\overline{A^{(2k)}} \subseteq \frac{m_k}{m_{k+1}}\frac{\overline{A}}{m_k} = \frac{\overline{A}}{m_{k+1}}$. Again, by Part 1 of Theorem 3.6 one has $\overline{A^{(2k+2)}} = \overline{A^{(2k+1)}} = \frac{\overline{A}}{m_{k+1}}$.

The inclusion $\frac{\overline{A}}{m_k} \subseteq \frac{\overline{A}}{\Delta^k}$ follows from Proposition 3.7. □

We will now assume that the non-singular matrix $\mathcal{A}$ is stable. Proposition 3.9 shows that the intermediate matrices remain both non-singular and stable, and Theorem 3.10 states that at the end we obtain a near-identity matrix.

Proposition 3.9. *Let $\mathcal{A} = [\alpha_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{E})$ be a reduced, non-singular, stable matrix, which is properly arranged with respect to a matrix of representatives P. Let $1 \le q \le 2n$. Then*
(1) $\Delta^{(q)}$ *is zeroless.*
(2) $\oslash\Delta < \Delta^{(q)} \subset \pounds$.
(3) $\overline{A^{(q)}} \subseteq \oslash\Delta^{(q)} \subseteq \oslash$.
(4) $\mathcal{A}^{(q)}$ *is limited, non-singular and stable.*

Proof. (1). Let $q = 2k$ or $q = 2k - 1$ with $1 \le k \le n$. By Lemma 3.4 one has $|d^{(q)}| = \left|\frac{d}{m_k}\right|$. Because the matrix is non-singular and stable, it holds that $\overline{A} \subseteq \oslash\Delta < |d|$, and because it is also reduced, it follows from Proposition 3.8 that $\overline{A^{(q)}} \subseteq \frac{\overline{A}}{m_k}$. Hence, $|d^{(q)}| > \overline{A^{(q)}}$. Also $D^{(q)} \subseteq \overline{A^{(q)}}$ by Propositions 3.2 and 2.17. Hence, $\Delta^{(q)}$ is zeroless.

(2). We show first that $D^{(q)} \subseteq \oslash$. Indeed, suppose $\oslash \subset D^{(q)}$. Then $\pounds \subseteq D^{(q)}$. By Propositions 2.17 and 3.2, it holds that $d^{(q)}$ is limited. This implies that $\Delta^{(q)}$ is a neutrix, in contradiction to Part 1. Hence, $D^{(q)} \subseteq \oslash$, which implies that $\Delta^{(q)} = d^{(q)} + D^{(q)} \subset \pounds$. It also follows from Part 1 that $\Delta^{(q)} \subseteq (1 + \oslash)d^{(q)}$. Now $d^{(q)} > \oslash\Delta$ by Lemma 3.4, hence also $\Delta^{(q)} > \oslash\Delta$.

(3). Let $1 \le q \le 2n$. Then $q = 2k$ or $q = 2k - 1$ with $1 \le k \le n$. By Proposition 3.8, the stability of the matrix $\mathcal{A}$, Lemma 3.4 and Part 2, one has

$$\overline{A^{(q)}} \subseteq \frac{\overline{A}}{m_k} \subseteq \frac{\oslash d}{m_k} = \oslash d^{(q)} = \oslash\Delta^{(q)} \subseteq \oslash.$$

(4). By Proposition 3.2 the matrix $A^{(q)}$ is limited. By Part 1 the matrix $\mathcal{A}^{(q)}$ is non-singular. Then $\mathcal{A}^{(q)}$ is stable by Part 3. ☐

Theorem 3.10. *Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ be a reduced, non-singular, stable matrix, which is properly arranged with respect to a matrix of representatives P. Then $\mathcal{G}^P\mathcal{A}$ is a near-identity matrix.*

Proof. Let A be the associated neutricial matrix of $\mathcal{A}$. By [17, Theorem 3.8] we have $\mathcal{G}^P\mathcal{A} = \mathcal{G}^P P + \mathcal{G}^P A = I + A'$, where $A' = [A'_{ij}]_{n\times n}$ is a neutricial matrix. By Part 3 of Proposition 3.9 one has $A' \subseteq [\oslash]_{n\times n}$. Hence, $\mathcal{G}^P\mathcal{A}$ is a near-identity matrix. ☐

We consider now the effect of the Gauss-Jordan procedure on the right-hand side of the system $\mathcal{A}|\mathcal{B}$. Proposition 3.11 gives bounds for the neutrices of the right-hand side, which are similar to the bounds for the neutrices of the intermediate matrices given in Proposition 3.8. It has an analogous proof.

Proposition 3.11. *Consider the system $\mathcal{A}|\mathcal{B}$, where $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ is a reduced, non-singular matrix, which is properly arranged with respect to a matrix of representatives P. Then for all k such that $1 \le k \le n$,*

$$B^{(2k)} = B^{(2k-1)} \subseteq \frac{B}{m_k} \subseteq \frac{B}{\Delta^k}.$$

If Δ is not an absorber of B, the neutrix part of the right-hand side is even invariant under the Gauss-Jordan procedure. The invariance follows from Proposition 3.12 and Theorem 3.13 and is also valid for the inverse Gauss-Jordan procedure. For the inverse procedure applied to the flexible system $\mathcal{G}^P\mathcal{A}|\mathcal{G}^P\mathcal{B}$ we define for $1 \le q \le 2n$

$$[B]^{(-q)} = ((\mathcal{G}^P_q)^{-1}((\mathcal{G}^P_{q+1})^{-1}\cdots((\mathcal{G}^P_{2n})^{-1}[B])))$$

and

$$(\mathcal{G}^P)^{-1}[B] = ((\mathcal{G}^P_1)^{-1}((\mathcal{G}^P_2)^{-1}\cdots((\mathcal{G}^P_{2n})^{-1}[B]))).$$

Proposition 3.12. *Suppose that the flexible system $\mathcal{A}|\mathcal{B}$ is properly arranged with respect to a representative matrix P and Δ is not an absorber of B. Then for all k such that* $1 \le k \le n-1$

$$\frac{m_{k+1}}{m_k}B = \frac{m_k}{m_{k+1}}B = B.$$

Proof. Let $1 \le k \le n-1$. By formula (8) it holds that $\oslash\Delta < \left|\frac{m_{k+1}}{m_k}\right| \in \pounds$. The fact that Δ is not an absorber of B and Proposition 2.6(6) imply that $\frac{m_{k+1}}{m_k}B = B$. It follows that $\frac{m_k}{m_{k+1}}B = B$ for $1 \le k \le n-1$. □

Theorem 3.13. *Suppose that the flexible system $\mathcal{A}|\mathcal{B}$ is properly arranged with respect to a representative matrix P and Δ is not an absorber of B. Then for all q such that $1 \le q \le 2n$ one has $[B]^{(q)} = [B]$ and $[B]^{(-q)} = [B]$. In particular, $\mathcal{G}^P[B] = [B]$ and $(\mathcal{G}^P)^{-1}[B] = [B]$.*

Proof. The proof is by External induction. Because $a_{11} = 1$, we have $[B]^{(1)} = \mathcal{G}_1^P[B] = I[B] = [B]$.

As for the induction step, let $q < 2n$ and suppose that $[B]^{(q)} = [B]$. We consider two cases.

Case 1: $q+1 = 2k+1$ for some $k \in \{1,\dots,n-1\}$. By the induction hypothesis and Proposition 3.12 we have

$$[B]^{(q+1)} = \mathcal{G}_{q+1}^P[B]^{(q)} = \mathcal{G}_{2k+1}^P[B] = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{m_k}{m_{k+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}.$$

Case 2: $q+1 = 2k+2$ for some $k \in \{1,\dots,n-1\}$. By Theorem 3.6(1) all entries of the matrix $\mathcal{G}_{2k+2}^P$ are limited, and the elements of its diagonal are equal to 1. Then it follows from Case 1 that

$$[B]^{(q+1)} = \mathcal{G}_{q+1}^P[B]^{(q)} = \mathcal{G}_{2k+2}^P[B]^{(2k+1)} = \mathcal{G}_{2k+2}^P[B] = [B].$$

In particular, $\mathcal{G}^P[B] = [B]^{(2n)}$. This proves the theorem for the Gauss-Jordan procedure $\mathcal{G}^P$. The proof for the inverse procedure is similar. □

Corollary 3.14. *Suppose that the flexible system $\mathcal{A}|\mathcal{B}$ is properly arranged with respect to a representative matrix P and Δ is not an absorber of B. Then*

$$(\mathcal{G}^P)^{-1}(\mathcal{G}^P\mathcal{B}) = \mathcal{B}.$$

Proof. Let $\mathcal{B} = b + B$. By [17, Theorem 3.8.1] and Theorem 3.13,

$$\begin{aligned}(\mathcal{G}^P)^{-1}\mathcal{G}^P\mathcal{B} &= (\mathcal{G}^P)^{-1}\mathcal{G}^P(b+B) = (\mathcal{G}^P)^{-1}(\mathcal{G}^Pb + \mathcal{G}^PB) \\ &= (\mathcal{G}^P)^{-1}(\mathcal{G}^Pb + B) = (\mathcal{G}^P)^{-1}\mathcal{G}^Pb + (\mathcal{G}^P)^{-1}B \\ &= ((\mathcal{G}^P)^{-1}\mathcal{G}^P)b + B = b + B = \mathcal{B}.\end{aligned}$$

□

To be able to prove that the Gauss-Jordan operations respect the stability property, we need one further property of the neutrix part of the right-hand side of stable systems. Not only it is invariant by multiplication by Δ, it is also invariant by dividing, and the same is true for the determinants $\Delta^{(q)}$.

Proposition 3.15. *Consider the system $\mathcal{A}|\mathcal{B}$. If Δ is not an absorber of B, it holds that*

$$\Delta B = \frac{B}{\Delta} = B. \tag{10}$$

Moreover, if the system is stable and $\mathcal{A}$ is properly arranged with respect to some representative matrix P, for $1 \le q \le 2n$ the determinant $\Delta^{(q)}$ is not an absorber of B, and

$$\Delta^{(q)}B = \frac{B}{\Delta^{(q)}} = B. \tag{11}$$

Proof. It follows from the fact that Δ is zeroless and Proposition 2.17 that $\oslash\Delta < \Delta \subset \pounds$. Also Δ is not an absorber of B. Then (10) follows from Proposition 2.6(7). It follows from Proposition 3.9(2) that $\oslash\Delta < \Delta^{(q)} \subset \pounds$. Then also $\Delta^{(q)}$ is not an absorber of B, hence (11) holds by Proposition 2.6(7). □

Proof of Theorem 2.35. (1). Assume that the system $\mathcal{A}|\mathcal{B}$ is stable. Let $1 \le q \le 2n$. By Proposition 3.9(4) the matrix $\mathcal{A}^{(q)}$ is stable. By Theorem 3.13 the system $\mathcal{A}^{(q)}|\mathcal{B}^{(q)}$ is uniform with $[B]^{(q)} = [B]$. Then $\Delta^{(q)}$ is not an absorber of $[B]^{(q)}$ by Proposition 3.15. We still need to show that

$$R(\mathcal{A}^{(q)}) \subseteq \mathrm{R}(\mathcal{B}^{(q)}).$$

Observe that $R(\mathcal{A}^{(q)})$ is well-defined, because $\Delta^{(q)}$ is zeroless by Proposition 3.9(1).

We show first that for $1 \le q \le 2n$

$$\overline{A^{(q)}}\ \overline{\beta^{(q)}} \subseteq B. \tag{12}$$

In order to derive (12), we show by External induction that for $0 \le q \le 2n$ and $1 \le i, j \le n$

$$A_{ij}^{(q)}\overline{\beta}^{(q)} \subseteq B. \tag{13}$$

For $q = 0$, we derive from stability, (5) and (10) that

$$A_{ij}^{(0)}\overline{\beta^{(0)}} \subseteq \overline{A}\,\overline{\beta} \subseteq \Delta R(\mathcal{A})\overline{\beta} \subseteq \Delta R(\mathcal{B})\overline{\beta} \subseteq \Delta B = B.$$

Assuming that formula (13) holds for $q < 2n$, we will prove it for $q+1$. Because $\overline{\beta^{(q+1)}} = \beta_p^{(q+1)}$ for some $p \in \{1, \ldots, n\}$,

$$|\overline{\beta^{(q+1)}}| = |\beta_p^{(q+1)}| = \left|\sum_{j=1}^{n} g_{pj}^{(q+1)}\beta_j^{(q)}\right| \le \sum_{j=1}^{n}|g_{pj}^{(q+1)}||\beta_j^{(q)}| \le \sum_{j=1}^{n}|g_{pj}^{(q+1)}||\overline{\beta^{(q)}}|.$$

Also

$$A_{ij}^{(q+1)} = g_{i1}^{(q+1)}A_{1j}^{(q)} + \cdots + g_{in}^{(q+1)}A_{nj}^{(q)}. \tag{14}$$

If $q+1 = 2k+2$ for some $k \in \{1, \ldots, n-1\}$, by Theorem 3.6 and the induction hypothesis one has

$$\begin{aligned} A_{ij}^{(q+1)}\overline{\beta^{(q+1)}} &\subseteq (g_{i1}^{(q+1)}A_{1j}^{(q)} + \cdots + g_{in}^{(q+1)}A_{nj}^{(q)})\left(\sum_{j=1}^{n}|g_{ij}^{(q+1)}||\overline{\beta^{(q)}}|\right) \\ &= (g_{i1}^{(q+1)}A_{1j}^{(q)}\overline{\beta^{(q)}} + \cdots + g_{in}^{(q+1)}A_{nj}^{(q)}\overline{\beta^{(q)}})\left(\sum_{j=1}^{n}|g_{ij}^{(q+1)}|\right) \\ &\subseteq (\pounds B + \cdots + \pounds B)\pounds \subseteq B. \end{aligned}$$

If $q+1 = 2k+1$ for some $k \in \{1, \ldots, n-1\}$, we consider separately the cases $i \ne k+1$ and $i = k+1$.

Case 1: For $i \ne k+1$ and $1 \le i \le n$, the row $g_i^{(q+1)}$ is a unit vector, so the neutrices of the ith row of $\mathcal{A}^{(q+1)}$ satisfy $A_{ij}^{(q+1)} = A_{ij}^{(q)}$ for $1 \le j \le n$. Also

$$\beta^{(q+1)} = \left(\beta_1^{(q)}, \ldots, \beta_k^{(q)}, \frac{m_k}{m_{k+1}}\beta_{k+1}^{(q)}, \beta_{k+2}^{(q)}, \ldots, \beta_n^{(q)}\right).$$

If $\overline{\beta^{(q+1)}} = \beta_s^{(q)}$ for some $s \in \{1, \ldots, n\} \setminus \{k+1\}$, for $i \ne k+1$, $1 \le i \le n$ and $1 \le j \le n$ one has by the induction hypothesis

$$A_{ij}^{(q+1)}\overline{\beta^{(q+1)}} = A_{ij}^{(q)}\beta_s^{(q)} \subseteq A_{ij}^{(q)}\overline{\beta^{(q)}} \subseteq B.$$

If $\overline{\beta^{(q+1)}} = \frac{m_k}{m_{k+1}}\beta_{k+1}^{(q)}$, then for $i \neq k+1$, $1 \leq i \leq n$ and $1 \leq j \leq n$ it follows from the induction hypothesis and Proposition 3.12 that

$$A_{ij}^{(q+1)}\overline{\beta^{(q+1)}} = A_{ij}^{(q)}\frac{m_k}{m_{k+1}}\beta_{k+1}^{(q)} \subseteq \frac{m_k}{m_{k+1}}B = B.$$

Case 2: For $i = k+1$, by formula (14) one has for $1 \leq j \leq n$

$$A_{k+1j}^{(q+1)} = A_{k+1j}^{(q)}\frac{m_k}{m_{k+1}}.$$

If $\overline{\beta^{(q+1)}} = \beta_s^{(q)}$ for some $s \in \{1,\dots,n\}\backslash\{k+1\}$, due to Proposition 3.12 one has for $1 \leq j \leq n$

$$A_{k+1j}^{(q+1)}\overline{\beta^{(q+1)}} = \frac{m_k}{m_{k+1}}A_{k+1j}^{(q)}\beta_s^{(q)} \subseteq \frac{m_k}{m_{k+1}}A_{k+1j}^{(q)}\overline{\beta^{(q)}} \subseteq \frac{m_k}{m_{k+1}}B = B.$$

If $\overline{\beta^{(q+1)}} = \frac{m_k}{m_{k+1}}\beta_{k+1}^{(q)}$, again using Proposition 3.12 we find for $1 \leq j \leq n$

$$A_{k+1j}^{(q+1)}\overline{\beta^{(q+1)}} = \frac{m_k}{m_{k+1}}A_{k+1j}^{(q)}\frac{m_k}{m_{k+1}}\beta_{k+1}^{(q)} \subseteq \left(\frac{m_k}{m_{k+1}}\right)^2 A_{k+1j}^{(q)}\overline{\beta^{(q)}} \subseteq \left(\frac{m_k}{m_{k+1}}\right)^2 B = B.$$

Combining, we see that (13) holds for all q such that $0 \leq q \leq 2n$.

Formula (12) follows directly from (13).

To finish the proof, we consider separately the cases that $\overline{\beta^{(q)}}$ is zeroless and that $\overline{\beta^{(q)}} = B$ is neutricial. If $\overline{\beta^{(q)}}$ is zeroless, by (12) and Proposition 3.15

$$\mathrm{R}(\mathcal{A}^{(q)}) = \frac{\overline{A^{(q)}}}{\Delta^{(q)}} \subseteq \frac{1}{\Delta^{(q)}}\frac{B}{\overline{\beta^{(q)}}} = \frac{B}{\overline{\beta^{(q)}}} = \mathrm{R}(\mathcal{B}^{(q)}).$$

If $\overline{\beta^{(q)}} = B$ is neutricial, formula (12) takes the form $\overline{A^{(q)}}B \subseteq B$. Then

$$\mathrm{R}(\mathcal{A}^{(q)})B = \overline{A^{(q)}}\frac{B}{\Delta^{(q)}} = \overline{A^{(q)}}B \subseteq B.$$

We conclude from Theorem 3.13 that $R(\mathcal{A}^{(q)}) \subseteq B : B = R([B^{(q)}]) = R(\mathcal{B}^{(q)})$.

(2). By setting $q = 2n$ we obtain from Part 1 that the final system $\mathcal{G}^P\mathcal{A}|\mathcal{G}^P\mathcal{B}$ is stable, while $\mathcal{G}^P A$ is a near-identity matrix by Theorem 3.10. □

3.2 Proof of Theorem 2.40

Here we extend the proof of Cramer's rule for flexible non-homogeneous systems given in [1] to homogeneous systems.

Let y be given by (6). Proposition 3.18 shows that the neutrix parts of the components of y are equal to the neutrix at the right-hand side B. Then the proof of Theorem 2.40 for homogeneous systems consists in showing that the solution is neutricial, with components equal to B.

Notation 3.16. Consider the system $\mathcal{A}|\mathcal{B}$ with $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ non-singular. Let $\Delta = \det(\mathcal{A}) \equiv d + D$ with $d = \det(P)$, where P is a representative matrix of $\mathcal{A}$. For $1 \leq i \leq n$, we let $M_i^P(b)$ be the matrix obtained from P by the substitution of the ith column by a representative vector b of $\mathcal{B}$.

Lemma 3.17. *Consider the system $\mathcal{A}|\mathcal{B}$. Then for $1 \leq j \leq n$*

(1) $|\det(M_j)| \leq 2n!|\overline{\beta}|$.

(2) $N(\det(M_j)) \subseteq \overline{\beta}\,\overline{A} + B$.

Proof. Let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$ and $\sigma \in S_n$. Put

$$\gamma_\sigma = \alpha_{\sigma(1)1} \ldots \alpha_{\sigma(j-1)j-1}\alpha_{\sigma(j+1)j+1} \ldots \alpha_{\sigma(n)n}.$$

Because the system is reduced,

$$|\gamma_\sigma| \le \overline{\alpha}^{n-1} \le (1+\oslash)^{n-1} = 1 + \oslash, \tag{15}$$

and, as a consequence of Proposition 2.6(6),

$$N(\gamma_\sigma) = N\left(\prod_{1\le k\le n, k\ne j} (a_{\sigma(k)k} + A_{\sigma(k)k})\right) \subseteq N(1+\overline{A})^{n-1} = \overline{A}. \tag{16}$$

(1). It follows from (15) that

$$|\det(M_j)| \le \sum_{\sigma\in S_n} |\gamma_\sigma \beta_{\sigma(j)}| \le \sum_{\sigma\in S_n} |(1+\oslash)|\overline{\beta}|| = n!(1+\oslash)|\overline{\beta}| = 2n!|\overline{\beta}|.$$

(2). It follows from (16) and (15) that

$$\begin{aligned} N(\det(M_j)) &= N\left(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\gamma_\sigma\beta_{\sigma(j)}\right) \\ &= \sum_{\sigma\in S_n} N(\gamma_\sigma\beta_{\sigma(j)}) \\ &= \sum_{\sigma\in S_n} (\beta_{\sigma(j)}N(\gamma_\sigma) + \gamma_\sigma N(\beta_{\sigma(j)})) \\ &\subseteq \sum_{\sigma\in S_n} (\overline{\beta}\overline{A} + (1+\oslash)B) = \overline{\beta}\overline{A} + B. \end{aligned}$$

□

Proposition 3.18. *Consider the system $\mathcal{A}|\mathcal{B}$. Assume that*
(1) $R(\mathcal{A}) \subseteq R(\mathcal{B})$
(2) Δ *is not an absorber of* B.

Then for $1 \le j \le n$

$$N\left(\frac{\det(M_j)}{\Delta}\right) = B.$$

As a consequence, if the system is homogeneous, for $1 \le j \le n$

$$\frac{\det(M_j)}{\Delta} = B.$$

Proof. Let $D = N(\Delta)$. It follows from Proposition 2.17 that $D \subseteq \overline{A}$. Then by Proposition 2.6(3) and Lemma 3.17 we have for $1 \le j \le n$

$$\begin{aligned} N\left(\frac{\det(M_j)}{\Delta}\right) &= \frac{1}{\Delta}N(\det(M_j)) + \det(M_j)N\left(\frac{1}{\Delta}\right) \\ &= \frac{1}{\Delta}N(\det(M_j)) + \det(M_j)\frac{D}{\Delta^2} \\ &\subseteq \frac{1}{\Delta}(\overline{\beta}\overline{A} + B) + 2n!\overline{\beta}\frac{D}{\Delta^2} \\ &\subseteq \frac{\overline{\beta}\overline{A}}{\Delta} + \frac{B}{\Delta} + \overline{\beta}\frac{\overline{A}}{\Delta^2}. \end{aligned} \tag{17}$$

From the condition $R(\mathcal{A}) \subseteq R(\mathcal{B})$ we derive both in the homogeneous and non-homogeneous case that $\overline{\beta}\frac{\overline{A}}{\Delta} \subseteq B$. Then we obtain from (17) and Proposition 3.15 that

$$N\left(\frac{\det(M_j)}{\Delta}\right) \subseteq \overline{\beta}\frac{\overline{A}}{\Delta} + \frac{B}{\Delta} + \frac{1}{\Delta}\left(\frac{\overline{\beta}}{\Delta}\overline{A}\right) \subseteq B + B + B/\Delta = B. \tag{18}$$

It follows from Proposition 4.8 of [17] that $|\Delta_{ij}| > \oslash\Delta$ for some $i \in \{1,\dots,n\}$. Because Δ is not an absorber of B, also Δ_{ij} is not an absorber of B. Hence, $B \subseteq B\Delta_{ij}$. It was shown in [17, Proposition 4.5] that the Laplace-expansion of a determinant takes the form of an inclusion. Because products containing a neutrix always have the same sign, we derive that

$$\begin{aligned} B &\subseteq B\Delta_{1j} + \cdots + B\Delta_{nj} \\ &\subseteq \det\begin{bmatrix} 1 + A_{11} & \cdots & \alpha_{1(j-1)} & B & \alpha_{1(j+1)} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{n(j-1)} & B & \alpha_{n(j+1)} & \cdots & \alpha_{nn} \end{bmatrix} \\ &\subseteq N\left(\det\begin{bmatrix} 1 + A_{11} & \cdots & \alpha_{1(j-1)} & b_1 + B & \alpha_{1(j+1)} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{n(j-1)} & b_n + B & \alpha_{n(j+1)} & \cdots & \alpha_{nn} \end{bmatrix}\right) \\ &= N(\det(M_j)). \end{aligned}$$

Then by Proposition 3.15

$$B = \frac{B}{\Delta} \subseteq \frac{N(\det(M_j))}{\Delta} \subseteq \frac{N(\det(M_j))}{\Delta} + \det(M_j)N\left(\frac{1}{\Delta}\right) = N\left(\frac{\det M_j}{\Delta}\right). \tag{19}$$

Combining (18) and (19), we conclude that $B = N\left(\frac{\det(M_j)}{\Delta}\right)$ for $1 \le j \le n$.

As a consequence, if the system is homogeneous, it holds that $\frac{\det(M_j)}{\Delta} = N\left(\frac{\det(M_j)}{\Delta}\right) = B$ for $1 \le j \le n$. □

Proof of Theorem 2.40. Let $x = (x_1, x_2, \dots, x_n)^T \in \gamma$. In order to show that x satisfies the system $\mathcal{A}|\mathcal{B}$, assume first that the system is not homogeneous. By Theorem 4.4 of [1] the external vector γ given by (6) is the solution of the system $\mathcal{A}|\mathcal{B}$, hence x satisfies $\mathcal{A}|\mathcal{B}$ by Remark 2.23. Second, assume that the system $\mathcal{A}|\mathcal{B}$ is homogeneous. Then $\gamma = (B, B, \dots, B)^T$ by Proposition 3.18. By direct verification we see that $\mathcal{A}\gamma \subseteq \mathcal{B}$. Again x is a solution of the system $\mathcal{A}|\mathcal{B}$ by Remark 2.23. Hence, $\gamma \subseteq S$.

Suppose now that $x \in S$. Let $P = [a_{ij}]_{n\times n}$ be a representative matrix for $\mathcal{A}$. Then for $1 \le i \le n$ there exists $b_i \in \beta_i$ such that

$$\begin{cases} a_{11}x_1 + & \cdots & +a_{1n}x_n & = & b_1 \\ \vdots & \ddots & \vdots & & \vdots \\ a_{n1}x_1 + & \cdots & +a_{nn}x_n & = & b_n. \end{cases}$$

Let $b = (b_1, \dots, b_n)^T$. By Cramer's rule, one has $x_i = \frac{\det(M_i^P(b))}{d} \in \frac{\det(M_i)}{\Delta}$ for $1 \le i \le n$. So $x \in \gamma$, hence $S \subseteq \gamma$. Combining, we conclude that $S = \gamma$. □

3.3 Proof of Theorem 2.38

In order to prove that the solution of a system $\mathcal{A}|\mathcal{B}$ is equal to the solution found at the end of the Gauss-Jordan procedure, we use some (sub)associative properties of multiplication of matrices, which were shown in [17]. Then it is also equal to the Cramer solution, due to Theorem 2.40.

Proof of Theorem 2.38. Let P be a properly arranged matrix of representatives of $\mathcal{A}$. Let $x \in \mathbb{R}^n$. Applying [17, Corollary 3.14.1] we derive with the help of external induction that

$$(\mathcal{G}^P\mathcal{A})x = \mathcal{G}^P(\mathcal{A}x). \tag{20}$$

(1). Let $x \in S$. Then $\mathcal{A}x \subseteq \mathcal{B}$. Now $\mathcal{G}^P(\mathcal{A}x) \subseteq \mathcal{G}^P\mathcal{B}$ by Proposition 2.19. Then $(\mathcal{G}^P\mathcal{A})x \subseteq \mathcal{G}^P\mathcal{B}$ by (20), meaning that $x \in G^P$. We conclude that $S \subseteq G^P$.

(2). Let $x \in G^P$. Then $(\mathcal{G}^P\mathcal{A})x \subseteq \mathcal{G}^P\mathcal{B}$. Using [17, Proposition 3.12.1], (20), Proposition 2.19 and Corollary 3.14, we derive that

$$\begin{aligned}\mathcal{A}x = I(\mathcal{A}x) &= ((\mathcal{G}^P)^{-1}\mathcal{G}^P)(\mathcal{A}x)\\ &\subseteq (\mathcal{G}^P)^{-1}(\mathcal{G}^P(\mathcal{A}x)) = (\mathcal{G}^P)^{-1}((\mathcal{G}^P(\mathcal{A}))x)\\ &\subseteq (\mathcal{G}^P)^{-1}\mathcal{G}^P\mathcal{B} = \mathcal{B}.\end{aligned}$$

So $x \in S$, hence $G^P \subseteq S$. Then it follows from Part (1) that $S = G^P$. Consequently, G^P does not depend on the choice of P, hence $G \equiv G^P$ is well-defined. We conclude that $S = G$. □

Corollary 3.19. *Consider the system $\mathcal{A}|\mathcal{B}$. Assume that $\mathcal{A}$ is properly arranged with respect to a representative matrix P and*
(1) $R(\mathcal{A}) \subseteq R(\mathcal{B})$
(2) *Δ is not an absorber of B.*

Then the Gauss-Jordan solution is equal to the Cramer solution.

Proof. The result follows directly from Theorems 2.38 and 2.40. □

3.4 Proof of the Main theorem

The proof of Theorem 1.1 is organized as follows. We show first that a system, which satisfies the stability condition of Definition 2.32(2) and is such that the coefficient matrix is a near-identity matrix, is simply solved by the right-hand side. Because the Gauss-Jordan procedure transforms a stable matrix into a stable near-identity matrix, we obtain as a corollary that the Gauss-Jordan procedure applied to the right-hand side is the Gauss-Jordan solution. Equality with the solution set S will follow from Theorem 2.38 and equality with the solution given by Cramer's rule from Theorem 2.40.

Proposition 3.20. *Suppose that the coefficient matrix of the system $\mathcal{A}|\mathcal{B}$ is a near-identity matrix and $R(\mathcal{A}) \subseteq R(\mathcal{B})$. Then $\mathcal{B}$ is the solution of the system.*

Proof. Let $b = (b_1, \dots, b_n)^T$ be a representative of $\mathcal{B}$. Because $\mathcal{A}$ is a near-identity matrix, the identity matrix I_n is a representative matrix of $\mathcal{A}$, so $\Delta = 1 + D$ with $D \subseteq \overline{A} \subseteq \oslash$. Hence, Δ is not an absorber of B. By Theorem 2.40 the Cramer solution $\gamma = (\gamma_1, \dots, \gamma_n)^T$ is well-defined, where $\gamma_i = \det(M_i)/\Delta$ and b_i is a representative of γ_i for $1 \le i \le n$. Also $N(\det(M_i)/\Delta) = B$ for $1 \le i \le n$ by Proposition 3.18. Hence,

$$\gamma_i = b_i + N\left(\frac{\det(M_i)}{\Delta}\right) = b_i + B$$

for $1 \le i \le n$, i.e., $\gamma = \mathcal{B}$. By Corollary 3.19 the vector $\mathcal{B}$ is also the Gauss-Jordan solution of the system. Then it is the solution by Theorem 2.38. □

Corollary 3.21. *Suppose that the system $\mathcal{A}|\mathcal{B}$ is stable and properly arranged with respect to a representative matrix P of $\mathcal{A}$. Then $\mathcal{G}^P\mathcal{B}$ is the Gauss-Jordan solution of $\mathcal{A}|\mathcal{B}$.*

Proof. By Theorem 3.10 it holds that $\mathcal{G}^P\mathcal{A}$ is a near-identity matrix. By Part 2 of Theorem 2.35 the system $\mathcal{G}^P\mathcal{A}|\mathcal{G}^P\mathcal{B}$ is stable. Then $R(\mathcal{G}^P\mathcal{A}) \subseteq R(\mathcal{G}^P\mathcal{B})$, so Proposition 3.20 implies that $\mathcal{G}^P\mathcal{B}$ is the solution of the system $\mathcal{G}^P\mathcal{A}|\mathcal{G}^P\mathcal{B}$, i.e., it is equal to the Gauss-Jordan solution G^P given by Definition 2.36. □

Proof of Theorem 1.1. Because the system is stable, Δ is not an absorber of B. By Theorem 2.38, the solution S is equal to the Gauss-Jordan solution G. Then $G = \mathcal{G}^P\mathcal{B}$ by Corollary 3.21. Also G is equal to the Cramer solution by Corollary 3.19, which takes the form (6) by Theorem 2.40. □

We end this subsection with an illustrative example. We verify first that the system is stable, then we show the Gauss-Jordan procedure in some detail, searching for neutrices instead of zeros, to see at the end that the right-hand side is the solution indeed. Observe that the solution is given in the form of truncated expansions.

Example 3.22. Consider the system

$$\begin{cases} (1+\varepsilon^2\oslash)x_1 + x_2 + (1+\varepsilon^3\pounds)x_3 \subseteq 1+\varepsilon\oslash \\ (1+\varepsilon^3\pounds)x_1 + \left(-\frac{1}{2}+\varepsilon^2\oslash\right)x_2 - \frac{1}{2}x_3 \subseteq -2+\varepsilon\oslash \\ \left(\frac{1}{2}\varepsilon+\varepsilon^3\oslash\right)x_1 + \frac{1}{2}x_2 + (1+\varepsilon^2\oslash)x_3 \subseteq \varepsilon+\varepsilon\oslash, \end{cases}$$

where ε is a positive infinitesimal. Let $\mathcal{A}$ be the matrix of coefficients and $\mathcal{B}$ be the right-hand side matrix.

The matrix $\mathcal{A}$ is reduced and non-singular, with $\Delta = \det\mathcal{A} = -\frac{3}{4} + \varepsilon^2\oslash$, which is zeroless. Let

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2}\varepsilon & \frac{1}{2} & 1 \end{bmatrix}.$$

Then P is a representative matrix of $\mathcal{A}$. One verifies that P is properly arranged, with $m_2 = m_{12}^{12} = -\frac{3}{2}$, $m_{13}^{12} = -\frac{3}{2}$, $m_{12}^{13} = \frac{1}{2} - \frac{1}{2}\varepsilon$, $m_{13}^{13} = 1 - \frac{1}{2}\varepsilon$. As a consequence, $\mathcal{A}$ is properly arranged. Because $R(\mathcal{A}) = \varepsilon^2\oslash \subseteq R(\mathcal{B}) = \varepsilon\oslash$ and $\Delta B = \left(-\frac{3}{4} + \varepsilon^2\oslash\right)\varepsilon\oslash = \varepsilon\oslash = B$, the system is stable.

The Gauss-Jordan procedure with respect to P leads to the following succession of stable systems:

$$\mathcal{A}|\mathcal{B} = \left[\begin{array}{ccc|c} 1+\varepsilon^2\oslash & 1 & 1+\varepsilon^3\pounds & 1+\varepsilon\oslash \\ 1+\varepsilon^3\pounds & -\frac{1}{2}+\varepsilon^2\oslash & -\frac{1}{2}\oslash & -2+\varepsilon\oslash \\ \frac{1}{2}\varepsilon+\varepsilon^3\oslash & \frac{1}{2} & 1+\varepsilon^2\oslash & \varepsilon+\varepsilon\oslash \end{array}\right]$$

$$\underset{\substack{L_2 - L_1 \\ L_3 - \frac{1}{2}\varepsilon L_1}}{\longrightarrow} \left[\begin{array}{ccc|c} 1+\varepsilon^2\oslash & 1 & 1+\varepsilon^3\pounds & 1+\varepsilon\oslash \\ \varepsilon^2\oslash & -\frac{3}{2}+\varepsilon^2\oslash & -\frac{3}{2}+\varepsilon^3\pounds & -3+\varepsilon\oslash \\ \varepsilon^3\oslash & \frac{1}{2}-\frac{1}{2}\varepsilon & 1-\frac{1}{2}\varepsilon+\varepsilon^2\oslash & \frac{1}{2}\varepsilon+\varepsilon\oslash \end{array}\right]$$

$$\underset{-\frac{2}{3}L_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1+\varepsilon^2\oslash & 1 & 1+\varepsilon^3\pounds & 1+\varepsilon\oslash \\ \varepsilon^2\oslash & 1+\varepsilon^2\oslash & 1+\varepsilon^3\pounds & 2+\varepsilon\oslash \\ \varepsilon^3\oslash & \frac{1}{2}-\frac{1}{2}\varepsilon & 1-\frac{1}{2}\varepsilon+\varepsilon^2\oslash & \frac{1}{2}\varepsilon+\varepsilon\oslash \end{array}\right]$$

$$\underset{\substack{L_1 - L_2 \\ L_3 - \left(\frac{1-\varepsilon}{2}\right)L_2}}{\longrightarrow} \left[\begin{array}{ccc|c} 1+\varepsilon^2\oslash & \varepsilon^2\oslash & \varepsilon^3\pounds & -1+\varepsilon\oslash \\ \varepsilon^2\oslash & 1+\varepsilon^2\oslash & 1+\varepsilon^3\pounds & 2+\varepsilon\oslash \\ \varepsilon^2\oslash & \varepsilon^2\oslash & \frac{1}{2}+\varepsilon^2\oslash & -1+\frac{3}{2}\varepsilon+\varepsilon\oslash \end{array}\right]$$

$$\underset{2L_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1+\varepsilon^2\oslash & \varepsilon^2\oslash & \varepsilon^3\pounds & -1+\varepsilon\oslash \\ \varepsilon^2\oslash & 1+\varepsilon^2\oslash & 1+\varepsilon^3\pounds & 2+\varepsilon\oslash \\ \varepsilon^2\oslash & \varepsilon^2\oslash & 1+\varepsilon^2\oslash & -2+3\varepsilon+\varepsilon\oslash \end{array}\right]$$

$$\underset{L_2 - L_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1+\varepsilon^2\oslash & \varepsilon^2\oslash & \varepsilon^3\pounds & -1+\varepsilon\oslash \\ \varepsilon^2\oslash & 1+\varepsilon^2\oslash & \varepsilon^2\oslash & 4-3\varepsilon+\varepsilon\oslash \\ \varepsilon^2\oslash & \varepsilon^2\oslash & 1+\varepsilon^2\oslash & -2+3\varepsilon+\varepsilon\oslash \end{array}\right] \equiv \mathcal{I}_3|S.$$

The matrix $\mathcal{I}_3$ is a near-identity matrix indeed, and its biggest neutrix $\varepsilon^2\oslash$ is contained in the neutrix $\varepsilon\oslash$ occurring in the right-hand side. Clearly $\det(\mathcal{I}_3) \subseteq 1 + \oslash$, so $R(\mathcal{I}_3) \subseteq R(S)$. By Proposition 3.20 the external vector

$$S \equiv \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} -1+\varepsilon\oslash \\ 4-3\varepsilon+\varepsilon\oslash \\ -2+3\varepsilon+\varepsilon\oslash \end{bmatrix}$$

solves $\mathcal{I}_3|S$, and then Theorem 1.1 implies that it solves the original system. It is straightforward to verify this by substitution and also to verify that Cramer's rule yields the same solution.

4 On the simplification of flexible systems

Van der Corput [10] observed that to obtain a formula we often calculate with unnecessary precision, e.g., too many terms of an expansion. His theory of functional neutrices was intended to develop a general approach to neglect superfluous terms. With this objective in mind, we present here a result on neglecting terms in the coefficient matrix of a flexible system, without affecting the solution set.

Flexible systems with the same right-hand side will be said to be equivalent if they have equal solutions. In particular, let $\mathcal{A}|\mathcal{B}$ be a system such that some of the entries of $\mathcal{A}$ are given in the form of expansions. Assume that $\mathcal{A}'$ is obtained from $\mathcal{A}$ by truncating the expansions in such a way that $\mathcal{A}'|\mathcal{B}$ is equivalent to $\mathcal{A}|\mathcal{B}$. Then we may solve as well the simplified system $\mathcal{A}'|\mathcal{B}$, neglecting the extra terms occurring in $\mathcal{A}$. If $\mathcal{A}|\mathcal{B}$ is stable, we will see that the simplification is justified if the neglected terms t of the expansions satisfy $t/\Delta \subseteq R(\mathcal{B})$. We may roughly interpret this as follows. Suppose the elements of the coefficient matrix of a system $\mathcal{A}|\mathcal{B}$ are obtained numerically, say by measurements or rounded-off calculations. Some decimals could possibly be omitted, if compared with the determinant they are small with respect to the relative imprecisions of the right-hand side, i.e., a coarser measurement would do as well, or a calculation with less digits.

Definition 4.1. Consider the system $\mathcal{A}|\mathcal{B}$. Let $\mathcal{A}' \in \mathcal{M}_n(\mathbb{E})$. The system $\mathcal{A}'|\mathcal{B}$ is said to be *equivalent* to $\mathcal{A}|\mathcal{B}$ if the solution of $\mathcal{A}'|\mathcal{B}$ is equal to the solution of $\mathcal{A}|\mathcal{B}$.

Theorem 4.2 gives conditions for such flexible systems to be equivalent.

Theorem 4.2. *Let $\mathcal{A}|\mathcal{B}$ be a stable system with solution $S = \mathcal{G}^P\mathcal{B}$, where $P = [a_{ij}]_{n\times n}$ is a reduced properly arranged representative matrix of $\mathcal{A}$. Let $Q = [q_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{R})$ be a reduced properly arranged matrix such that for $1 \le i, j \le n$*

$$q_{ij} - a_{ij} \in \overline{A}. \tag{21}$$

Let $\mathcal{A}' \equiv [\alpha'_{ij}]$ with $\alpha'_{ij} = q_{ij} + A'_{ij}$ and $A'_{ij} \subseteq \overline{A}$ for $1 \le i, j \le n$. Then $\mathcal{A}'|\mathcal{B}$ is a stable system equivalent to $\mathcal{A}|\mathcal{B}$, and $\mathcal{G}^P\mathcal{B} = \mathcal{G}^Q\mathcal{B}$.

We prove first two lemmas.

Lemma 4.3. *Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{E}) = [\alpha_{ij}]_{n\times n} = [a_{ij} + A_{ij}]_{n\times n}$ be a reduced, non-singular, stable matrix, properly arranged with respect to a reduced representative matrix $P = [a_{ij}]_{n\times n}$. Let $\mathcal{A}' \equiv [\alpha'_{ij}]_{n\times n}$ be defined by*

$$\alpha'_{ij} = a_{ij} + A'_{ij},$$

with $A'_{ij} \subseteq \overline{A}$ *for* $1 \le i, j \le n$. *Let* $\Delta' = \det(\mathcal{A}')$. *Then* $\Delta' \subseteq (1 + \oslash)\Delta$ *and the matrix* $\mathcal{A}'$ *is reduced, non-singular and stable, with* $R(\mathcal{A}') \subseteq R(\mathcal{A})$.

Proof. Because $\mathcal{A}$ is reduced and $\overline{A'} \subseteq \overline{A} \subseteq \oslash$, the matrix $\mathcal{A}'$ is also reduced. Let $d = \det(P)$. Then d and Δ' are limited. Because $A'_{ij} \subseteq \overline{A}$ for $1 \le i, j \le n$, and the matrix $\mathcal{A}$ is non-singular and stable, it holds that $\Delta' \subseteq d + \overline{A} \subseteq \Delta + \oslash\Delta = (1 + \oslash)\Delta$. This implies that Δ' is zeroless, hence $\mathcal{A}'$ is non-singular. In addition

$$\frac{\overline{\mathcal{A}'}}{\Delta'} \subseteq \frac{\overline{\mathcal{A}}}{(1 + \oslash)\Delta} = \frac{\overline{\mathcal{A}}}{\Delta} \subseteq \oslash.$$

Hence, $R(\mathcal{A}') \subseteq R(\mathcal{A})$ and $\mathcal{A}'$ is stable. □

Lemma 4.4. *Assume that the system* $\mathcal{A}|\mathcal{B}$ *is stable, and* $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ *is properly arranged with respect to a matrix of representatives* $P = [a_{ij}]_{n \times n}$. *Let* $\mathcal{A}' \equiv [\alpha'_{ij}]$ *with* $\alpha'_{ij} = a_{ij} + A'_{ij}$ *and* $A'_{ij} \subseteq \overline{A}$ *for* $1 \le i, j \le n$. *Then* $\mathcal{A}'|\mathcal{B}$ *is a stable system satisfying Convention* 2.37, *equivalent to* $\mathcal{A}|\mathcal{B}$.

Proof. By Lemma 4.3 the matrix $\mathcal{A}'$ is reduced and non-singular. Then $\mathcal{A}'|\mathcal{B}$ satisfies Convention 2.37. Again by Lemma 4.3 the matrix $\mathcal{A}'$ is stable, while $\Delta' \subseteq (1 + \oslash)\Delta$ and $R(\mathcal{A}') \subseteq R(\mathcal{A})$. Then $R(\mathcal{A}') \subseteq R(\mathcal{B})$ and Δ' is not an absorber of B. Hence, $\mathcal{A}'|\mathcal{B}$ is stable. By Corollary 3.21 both systems $\mathcal{A}|\mathcal{B}$ and $\mathcal{A}'|\mathcal{B}$ are solved by $\mathcal{G}^P\mathcal{B}$. Hence, the systems are equivalent. □

Proof of Theorem 4.2. Put $\mathcal{A}'' = Q + (\overline{A})_{n \times n}$. It follows from (21) that $\mathcal{A}'' = P + (\overline{A})_{n \times n}$, so by Lemma 4.3 the matrix $\mathcal{A}''$ is reduced, non-singular and stable. Then the system $\mathcal{A}''|\mathcal{B}$ satisfies Convention 2.37 and is stable. By Lemma 4.4 the systems $\mathcal{A}|\mathcal{B}$ and $\mathcal{A}''|\mathcal{B}$ are equivalent. The system $\mathcal{A}'|\mathcal{B}$ shares with the system $\mathcal{A}''|\mathcal{B}$ the representative matrix Q, hence by Lemma 4.4 it satisfies Convention 2.37 is stable and equivalent to $\mathcal{A}''|\mathcal{B}$. Hence, the systems $\mathcal{A}|\mathcal{B}$ and $\mathcal{A}'|\mathcal{B}$ are also equivalent. Then it follows from Corollary 3.21 that $\mathcal{G}^P\mathcal{B} = \mathcal{G}^Q\mathcal{B}$. □

Corollary 4.5. *Let* $\mathcal{A}|\mathcal{B}$ *be a stable system, where* $\mathcal{A} = P + A$, *with* $P = [a_{ij}]_{n \times n}$ *a reduced properly arranged representative matrix and* A *a neutricial matrix. Let* $Q = [q_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{R})$ *be a reduced properly arranged representative matrix of* $\mathcal{A}' \equiv P + (\overline{A})_{n \times n}$. *Then* $\mathcal{A}|\mathcal{B}$ *and* $Q|\mathcal{B}$ *are equivalent.*

Proof. The corollary follows from Theorem 4.2, by taking $A'_{ij} = 0$ for all $1 \le i, j \le n$. □

The corollary indicates that for stable systems we may neglect all terms and neutrices smaller than the biggest neutrix in the coefficient matrix, and solve instead for any real coefficient matrix lying within this range of imprecision. We will illustrate this with Example 4.6.

Example 4.6. Let $\varepsilon > 0$ be infinitesimal. Consider the reduced flexible system $\mathcal{A}|\mathcal{B}$ given by

$$\begin{cases} (1 + \varepsilon £)x_1 + (1 - \varepsilon)x_2 + (\frac{1}{2} + 2\varepsilon^2)x_3 + \frac{1}{2}x_4 \subseteq -1 + \varepsilon £ \\ (-1 + 3\varepsilon)x_1 + x_2 + \left(\frac{1}{2} + \varepsilon^2 + \varepsilon^2\oslash\right)x_3 + \frac{1}{2}x_4 \subseteq \varepsilon £ \\ x_2 - \frac{1}{2}x_3 + (1 - 3\varepsilon^2 + \varepsilon^2\oslash)x_4 \subseteq -\frac{1}{2} + \varepsilon £ \\ \left(\frac{1}{2} + \varepsilon + \varepsilon\oslash\right)x_1 + (1 + \varepsilon £)x_3 + (1 + \varepsilon\oslash)x_4 \subseteq 2 + \varepsilon £ \end{cases} . \tag{22}$$

The matrix

$$P = \begin{bmatrix} 1 & 1-\varepsilon & 1/2+2\varepsilon^2 & 1/2 \\ -1+3\varepsilon & 1 & 1/2+\varepsilon^2 & 1/2 \\ 0 & 1 & -1/2 & 1-3\varepsilon^2 \\ 1/2+\varepsilon & 0 & 1 & 1 \end{bmatrix}$$

is a representative matrix of $\mathcal{A}$. One verifies that P is non-singular, reduced and properly arranged, with $\det(P) \in -3 + \pounds\varepsilon$ zeroless, $m_1 = 1, m_2 \in 2 - 2\varepsilon + \oslash\varepsilon$ and $m_3 \in 2 - (5/2)\varepsilon + \oslash\varepsilon$. Also $R(\mathcal{A}) = \overline{A}/\Delta = \varepsilon\pounds, R(\mathcal{B}) = B/\overline{\beta} = \varepsilon\pounds$ and $\Delta B = \varepsilon\pounds = B$. Hence, $\overline{A} = \varepsilon\pounds \subset \oslash = \oslash\Delta$, $R(\mathcal{A}) \subseteq R(\mathcal{B})$ and Δ is not an absorber of B. As a consequence the system $\mathcal{A}|\mathcal{B}$ is stable.

Let

$$Q = \begin{bmatrix} 1 & 1 & 1/2 & 1/2 \\ -1 & 1 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1 \\ 1/2 & 0 & 1 & 1 \end{bmatrix}$$

The matrix Q is reduced and non-singular, with determinant $d \equiv \det(Q) = -3$. A straightforward calculation shows that Q is properly arranged, with $m_2 = 2$ and $m_3 = -2$. The entries of Q and P differ for at most a limited multiple of ε, which is contained in $\overline{A} = \pounds\varepsilon$, while $\overline{A'} = 0 \subset \overline{A}$.

Applying the usual Gauss-Jordan procedure we derive that

$$\mathcal{X} = \mathcal{G}^Q\mathcal{B} = \begin{bmatrix} -1/2+\varepsilon\pounds \\ -13/8+\varepsilon\pounds \\ 3/4+\varepsilon\pounds \\ 3/2+\varepsilon\pounds \end{bmatrix}$$

is the Gauss-Jordan solution of the system $Q|\mathcal{B}$. By Corollary 4.5, it is also the solution of (22).

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