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Some New Oscillation Theorems for Second Order Difference Equations with Mixed Neutral Terms

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Abstract

Some new sufficient conditions are established for the oscillation of all solutions of the second order neutral difference equation of the form

$$\Delta\left(r_n\Delta\left(x_n+ax_{n-\tau}+bx_{n+\sigma}\right)\right)+p_nx^{\alpha}_{n-k}+q_nx^{\beta}_{n+m}=0,\quad n\geq n_0,$$

where $\sum_{n=n_0}^{\infty}1/r_n<\infty$. The results obtained here are new and further improve and complement some known results in the literature. Examples are provided to illustrate the main results.

Keywords: Second order; neutral type difference equation; mixed neutral term; oscillation.
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1 Introduction

This paper deals with the oscillatory behavior of solutions of second order neutral difference equation of the form

$$\Delta\left(r_n\Delta\left(x_n + ax_{n-\tau} + bx_{n+\sigma}\right)\right) + p_n x_{n-k}^{\alpha} + q_n x_{n+m}^{\beta} = 0, \quad n \geq n_0 \tag{1.1}$$

where n_0 is a positive integer, Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, and α and β are ratios of odd positive integers. Further, we assume the following conditions without further mention:

(C1) $\{r_n\}$ is a sequence of positive real numbers for all $n \geq n_0$ with $\sum_{n=n_0}^{\infty} 1/r_n < \infty$;

(C2) a and b are nonnegative real constants;

(C3) $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences, and not identically zero for many values of n;

(C4) τ, σ, k and m are nonnegative integers.

By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta_1$ where $\theta_1 = \max\{\tau, k\}$ and satisfies equation (1.1) for all $n \geq n_0$. Let $\theta_2 = \max\{\sigma, m\}$. Clearly, if the initial conditions $x_n = \phi_n$ for all $n \in [n_0 - \theta_1, n_0 + \theta_2 - 1]$ are given, then equation (1.1) has a unique solution satisfying the initial conditions. A solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

In recent years, there has been an increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of neutral type difference equations, see for example [1], [2], [3], [4], [5], [6], [7], [17] and the references cited therein. In [8], [9], [10], [11], [12], [13], [14], [15] and [16] the authors considered equation of the form (1.1), and established some sufficient conditions for the oscillation of all solutions when $\sum_{n=n_0}^{\infty} 1/r_n = \infty$. This is because such equations have applications in science and technology [18].

However when $b = 0$ and either $p_n = 0$ or $q_n = 0$, there are some results available in the literature dealing with the oscillatory behavior of equation (1.1) with $\sum_{n=n_0}^{\infty} 1/r_n < \infty$, see for example [1], [2], [3] and [6], and the references cited therein. In particular in [17], the authors considered equation (1.1) under the condition (C1), and obtained criteria which imply that all solutions of equation (1.1) are either oscillatory or tend to zero as $n \to \infty$.

Equation (1.1) may be considered as a discrete analogue of the continuous equation

$$(r(t)(z(t))')' + p(t)x(t-k)^{\alpha} + q(t)x(t+m)^{\beta} = 0 \tag{1.2}$$

where $z(t) = x(t) + ax(t-\tau) + bx(t+\sigma)$. The oscillatory and asymptotic behavior of solutions of equation (1.2) is studied in [19], [20] and [6] under various conditions on the known functions. However we have found some results on the oscillatory behavior of solutions of equation (1.2) for the particular case $b = 0$, $q(t) = 0$ and $\int_{t_0}^{\infty} 1/r(t)dt < \infty$, see for example [21], and the references contained therein.

Therefore the natural question arises whether all solutions of equation (1.1) are oscillatory if the hypothesis (C1) is satisfied. Our purpose in this paper is to give answer to this question in a affirmative way by obtaining some new sufficient conditions which ensure that all solutions of equation (1.1) are oscillatory. In Section 2, we present some preliminary lemmas and in Section 3, we obtain some new oscillation criteria for equation (1.1). Section 4 contains some examples to illustrate the main results. Thus, the results obtained in this paper are new and further improve and complement to those established in [2], [3], [6] and [17].

2 Some Preliminary Lemmas

In this section we present some lemmas which will be needed to prove the main results. For the sake of convenience, we define the following notations:

$$\begin{aligned} z_n &= x_n + ax_{n-\tau} + bx_{n+\sigma}, \\ P_n &= min\left\{p_n, p_{n-\tau}, p_{n+\sigma}\right\}, \\ Q_n &= min\left\{q_n, q_{n-\tau}, q_{n+\sigma}\right\}, \\ R_n &= \sum_{s=n_0}^{n-1} \frac{1}{r_s} \quad \text{, and} \quad A_n = \sum_{s=n}^{\infty} \frac{1}{r_s}. \end{aligned}$$

We begin with the following lemma.

Lemma 2.1. *Let $\{x_n\}$ be an eventually positive solution of equation (1.1). Then one of the following two cases holds for all sufficiently large n:*

(I) $z_n > 0, \quad r_n\Delta z_n > 0, \quad \Delta\left(r_n\Delta z_n\right) \le 0$;

(II) $z_n > 0, \quad r_n\Delta z_n < 0, \quad \Delta\left(r_n\Delta z_n\right) \le 0.$

Proof. The proof can be found in [5]. □

Lemma 2.2. *Let $A \ge 0,$ and $B \ge 0$. Then for $\gamma \ge 1$,we have* $A^\gamma + B^\gamma \ge \frac{1}{2^{\gamma-1}}\left(A+B\right)^\gamma$.

Proof. The proof can be found in Chapter III, page 70 of [22]. □

Lemma 2.3. *Let $\{x_n\}$ be a positive solution of equation (1.1) satisfying case (I) of Lemma 2.1. Then*

$$z_n \ge R_n r_n \Delta z_n \tag{2.1}$$

and $\left\{\frac{z_n}{R_n}\right\}$ is nonincreasing for all $n \ge N$.

Proof. The proof of (2.1) can be found in [13]. Further $\Delta(\frac{z_n}{R_n}) = -\frac{1}{r_n}[\frac{z_n - r_n R_n \Delta z_n}{R_n R_{n+1}}] \le 0$ since $\Delta R_n = \frac{1}{r_n}$. The proof is now completed. □

Lemma 2.4. *Let $\{x_n\}$ be a positive solution of equation (1.1) satisfying case (II) of Lemma 2.1. Then*

$$z_n \ge -A_n r_n \Delta z_n \tag{2.2}$$

for all $n \ge N$.

Proof. Since $r_n\Delta z_n$ is nonincreasing, we have

$$r_s\Delta z_s \le r_n \Delta z_n, \quad s \ge n \ge N.$$

Dividing the last inequality by r_s and summing it from n to l, we obtain

$$z_{l+1} \le z_n + r_n\Delta z_n \sum_{s=n}^{l} \frac{1}{r_s}, \quad l \ge n \ge N.$$

Letting $l \to \infty$, we have

$$0 \le z_n + A_n r_n \Delta z_n, \quad n \ge N.$$

The proof is now completed. □

3 Oscillation Theorems

In this section, we obtain some new sufficient conditions for the oscillation of all solutions of equation (1.1). We begin with the following theorem.

Theorem 3.1. *Assume that* $1 \leq \beta \leq \alpha$, $0 \leq a, b \leq 1, k - \tau \geq 1$ *and* $m - \sigma \geq 0$. *If the difference inequality*

$$\Delta w_n + \frac{P_n}{4^{\alpha-1}} \frac{R_{n-k}^{\alpha}}{\left(1 + a^{\beta} + \frac{b^{\beta}}{2^{\beta-1}}\right)^{\alpha}} w_{n-k+\tau}^{\alpha} \leq 0, \quad n \geq n_0, \tag{3.1}$$

has no eventually positive nonincreasing solutions, and the difference inequality

$$\Delta w_n - \frac{Q_n}{4^{\beta-1}} \frac{A_{n+m}^{\beta}}{\left(1 + a^{\beta} + \frac{b^{\beta}}{2^{\beta-1}}\right)^{\beta}} w_{n+m-\sigma}^{\beta} \geq 0, \quad n \geq n_0, \tag{3.2}$$

has no eventually positive nondecreasing solutions, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x_{n-\theta_1} > 0$ for all $n \geq N_1 \geq n_0$, where N_1 is chosen so that both the cases of Lemma 2.1 hold for all $n \geq N_1$. From the equation (1.1), we have

$$\begin{aligned} &\Delta\left(r_n \Delta z_n\right) + p_n x_{n-k}^{\alpha} + q_n x_{n+m}^{\beta} + a^{\beta} \Delta\left(r_{n-\tau} \Delta z_{n-\tau}\right) + a^{\beta} p_{n-\tau} x_{n-k-\tau}^{\alpha} \\ &\quad + a^{\beta} q_{n-\tau} x_{n+m-\tau}^{\beta} + \frac{b^{\beta}}{2^{\beta-1}} \Delta\left(r_{n+\sigma} \Delta z_{n+\sigma}\right) + \frac{b^{\beta}}{2^{\beta-1}} p_{n+\sigma} x_{n-k+\sigma}^{\alpha} \\ &\quad + \frac{b^{\beta}}{2^{\beta-1}} q_{n+\sigma} x_{n+m+\sigma}^{\beta} = 0, \quad n \geq N_1. \end{aligned} \tag{3.3}$$

Since $\beta \leq \alpha$, $a \leq 1$ and $b \leq 1$, we have

$$\begin{aligned} &\Delta\left(r_n \Delta z_n\right) + a^{\beta} \Delta\left(r_{n-\tau} \Delta z_{n-\tau}\right) + \frac{b^{\beta}}{2^{\beta-1}} \Delta\left(r_{n+\sigma \Delta z_{n+\sigma}}\right) \\ &\quad + P_n \left[x_{n-k}^{\alpha} + a^{\alpha} x_{n-k-\tau}^{\alpha} + \frac{b^{\alpha}}{2^{\alpha-1}} x_{n-k+\sigma}^{\alpha}\right] \\ &\quad + Q_n \left[x_{n+m}^{\beta} + a^{\beta} x_{n+m-\tau}^{\beta} + \frac{b^{\beta}}{2^{\beta-1}} x_{n+m+\sigma}^{\beta}\right] \leq 0, \quad n \geq N_1. \end{aligned}$$

Using the Lemma 2.2, we obtain

$$\begin{aligned} &\Delta\left(r_n \Delta z_n + a^{\beta} r_{n-\tau} \Delta z_{n-\tau} + \frac{b^{\beta}}{2^{\beta-1}} r_{n+\sigma} \Delta z_{n+\sigma}\right) \\ &\quad + \frac{P_n}{4^{\alpha-1}} z_{n-k}^{\alpha} + \frac{Q_n}{4^{\beta-1}} z_{n+m}^{\beta} \leq 0, \quad n \geq N_1. \end{aligned} \tag{3.4}$$

Next we consider the two cases of Lemma 2.1.
Case (I). Assume that case (I) of Lemma 2.1 holds for all $n \geq N_1$. From (3.4) and Lemma 2.3, we obtain

$$\Delta\left(y_n + a^{\beta} y_{n-\tau} + \frac{b^{\beta}}{2^{\beta-1}} y_{n+\sigma}\right) + \frac{P_n}{4^{\alpha-1}} R_{n-k}^{\alpha} y_{n-k}^{\alpha} \leq 0, \tag{3.5}$$

where $y_n = r_n \Delta z_n$ and $y_n > 0$ is nonincreasing for all $n \geq N_1$. Now we define, $w_n = y_n + a^{\beta} y_{n-\tau} + \frac{b^{\beta}}{2^{\beta-1}} y_{n+\sigma}$, then $w_n > 0$, $n \geq N_1$, and

$$w_n \leq \left(1 + a^{\beta} + \frac{b^{\beta}}{2^{\beta-1}}\right) y_{n-\tau}, \quad n \geq N_1.$$

Using the last inequality in (3.5), we obtain that inequality (3.1) has an eventually positive nonincreasing solution, a contradiction.
Case (II). Assume that case (II) of Lemma 2.1 holds for all $n \geq N_1$. From (3.4) and Lemma 2.4, we have

$$\Delta\left(y_n + a^\beta y_{n-\tau} + \frac{b^\beta}{2^{\beta-1}} y_{n+\sigma}\right) - \frac{Q_n}{4^{\beta-1}} A^\beta_{n+m} y^\beta_{n+m} \geq 0, \tag{3.6}$$

where $y_n = -r_n \Delta z_n$ and $y_n > 0$ is nondecreasing for all $n \geq N_1$. Now we define

$$w_n = y_n + a^\beta y_{n-\tau} + \frac{b^\beta}{2^{\beta-1}} y_{n+\sigma} \leq \left(1 + a^\beta + \frac{b^\beta}{2^{\beta-1}}\right) y_{n+\sigma}, \quad n \geq N_1.$$

Using the last inequality in (3.6), we obtain that the inequality (3.2) has an eventually positive nondecreasing solution, a contradiction. This completes the proof. ☐

Corollary 3.2. *Assume* $0 \leq a, b \leq 1$, $\alpha = \beta = 1$, $k - \tau \geq 1$, *and* $m - \sigma \geq 0$ *hold. If*

$$\lim_{n\to\infty} \inf \sum_{s=n+\tau-k}^{n-1} P_s R_{s-k} > M \left(\frac{k-\tau}{1+k-\tau}\right)^{1+k-\tau}, \tag{3.7}$$

and

$$\lim_{n\to\infty} \inf \sum_{s=n}^{n+m-\sigma} Q_s A_{s+m} > M \left(\frac{m-\sigma}{1+m-\sigma}\right)^{1+m-\sigma} \tag{3.8}$$

where $M = (1 + a + b)$ *are satisfied, then every solution of equation (1.1) is oscillatory.*

Proof. The proof follows from Theorem 3.1 and Theorem 7.6.1 of [23]. ☐

Corollary 3.3. *Assume* $0 \leq a, b \leq 1, \beta = 1, \alpha > 1, k - \tau \geq 1$, *and* $m - \sigma \geq 0$. *If condition (3.8) holds and there exists a* $\lambda > \frac{1}{k-\tau} log\alpha$ *such that*

$$\lim_{n\to\infty} \inf \left[P_n R^\alpha_{n-k} exp(-e^{\lambda n})\right] > 0, \quad n \geq N$$

then every solution of equation (1.1) is oscillatory.

Proof. The proof follows from Theorem 3.1, and Theorem 2 of [24]. ☐

In the following we derive some new oscillation criteria for the equation (1.1) where $q_n = 0$ for all $n \geq n_0$. In this case the equation (1.1) takes the form

$$\Delta\left(r_n \Delta\left(x_n + a x_{n-\tau} + b x_{n+\sigma}\right)\right) + p_n x^\alpha_{n-k} = 0, \quad n \geq n_0. \tag{3.9}$$

Theorem 3.4. *Assume that* $0 < \alpha \leq 1$, $\left(1 - a - \frac{bR_{n+\sigma-k}}{R_{n-k}}\right) > 0$ *for all* $n \geq n_0$, *and there exists a positive nondecreasing real sequence* $\{\rho_n\}$ *such that for any constant* $M_1 > 0$

$$\sum_{n=n_0}^{\infty} \left[\rho_n p_n \left(1 - a - b\frac{R_{n+\sigma-k}}{R_{n-k}}\right)^\alpha - \frac{M_1^{1-\alpha} \Delta\rho_n}{R^\alpha_{n-k}}\right] = \infty. \tag{3.10}$$

If there exists a positive real sequence $\{\delta_n\}$ *such that*

$$\frac{\delta_n}{r_n R_n} + \Delta\delta_n \leq 0, \qquad \left(1 - a\frac{\delta_{n-\tau-k}}{\delta_{n-k}} - b\right) > 0, \tag{3.11}$$

and for any constant $M_2 > 0$

$$\sum_{n=n_0}^{\infty} \left[M_2^{\alpha-1} p_n A_{n+1} \left(1 - a\frac{\delta_{n-\tau-k}}{\delta_{n-k}} - b\right)^\alpha - \frac{1}{4 r_n A_{n+1}}\right] = \infty \tag{3.12}$$

then every solution of equation (3.9) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x_{n-\theta_1} > 0$ for all $n \geq N \geq n_0$, where N is chosen so that both the cases of Lemma 2.1 hold for all $n \geq N$. From the equation (3.9), we have

$$\Delta\left(r_n \Delta z_n\right) + p_n x_{n-k}^{\alpha} = 0, \quad n \geq N. \tag{3.13}$$

Case (I). Assume that case (I) of Lemma 2.1 holds for all $n \geq N$. From Lemma 2.3, we have $\{z_n/R_n\}$ is nonincreasing for all $n \geq N$. By the definition of z_n, we have

$$\begin{aligned} x_n &\geq z_n - a z_{n-\tau} - b z_{n+\sigma} \\ &\geq \left(1 - a - b\frac{R_{n+\sigma}}{R_n}\right) z_n \end{aligned} \tag{3.14}$$

where we have used $\{z_n\}$ is nondecreasing and $\{z_n/R_n\}$ is nonincreasing. Using (3.14) in (3.13), we obtain

$$\Delta\left(r_n \Delta z_n\right) + B_{n-k}^{\alpha} p_n z_{n-k}^{\alpha} \leq 0, \quad n \geq N \tag{3.15}$$

where $B_{n-k}^{\alpha} = \left(1 - a - b\frac{R_{n+\sigma-k}}{R_{n-k}}\right)^{\alpha}$. Define

$$w_n = \frac{\rho_n r_n \Delta z_n}{z_{n-k-1}^{\alpha}}, \quad n \geq N.$$

Then $w_n > 0$ and from (3.15), we have

$$\begin{aligned} \Delta w_n &\leq -\rho_n p_n B_{n-k}^{\alpha} + \Delta\rho_n \frac{r_n \Delta z_n}{z_{n-k}^{\alpha}} - \frac{\rho_n r_n \Delta z_n}{z_{n-k}^{\alpha} z_{n-1-k}^{\alpha}} \\ &\leq -\rho_n p_n B_{n-k}^{\alpha} + \frac{\Delta\rho_n r_n \Delta z_n}{R_{n-k}^{\alpha}\left(r_{n-k}\Delta z_{n-k}\right)^{\alpha}} \end{aligned}$$

where we have used (2.1). From the monotonicity of $\{r_n \Delta z_n\}$ and $0 < \alpha \leq 1$, we have from the last inequality

$$\Delta w_n \leq -\rho_n p_n B_{n-k}^{\alpha} + \frac{\Delta\rho_n}{R_{n-k}^{\alpha}} M_1^{1-\alpha}, \quad n \geq N, \tag{3.16}$$

where $M_1 = r_{N-k}\Delta z_{N-k}$. Summing the inequality (3.16) from N to $n-1$, we obtain

$$0 < w_n \leq w_{N_1} - \sum_{s=N}^{n-1}\left(\rho_s p_s B_{s-k}^{\alpha} + \frac{\Delta\rho_s}{R_{s-k}^{\alpha}} M_1^{1-\alpha}\right). \tag{3.17}$$

Letting $n \to \infty$ in (3.17), we obtain a contradiction to (3.10).
Case (II). Assume that case (II) of Lemma 2.1 holds for all $n \geq N$. Define

$$v_n = \frac{r_n \Delta z_n}{z_n}, \quad n \geq N. \tag{3.18}$$

Then $v_n < 0$ for $n \geq N$. From Lemma 2.4, we have

$$A_n r_n \frac{\Delta z_n}{z_n} \geq -1 \quad \textit{for all} \quad n \geq N. \tag{3.19}$$

By virtue of (3.18) and (3.19), we have

$$-1 \leq A_n v_n \leq 0, \quad n \geq N. \tag{3.20}$$

On the other hand, it follows from (3.19) that

$$\frac{\Delta z_n}{z_n} \geq -\frac{1}{A_n r_n}.$$

Then, we have

$$\Delta\left(z_n/\delta_n\right) \geq -\frac{z_n}{\delta_n\delta_{n+1}}\left[\frac{\delta_n}{A_n r_n}+\Delta\delta_n\right] \geq 0$$

and then $\{z_n/\delta_n\}$ is nondecreasing. Hence we obtain from the definition of z_n that

$$x_n \geq \left(1-a\frac{\delta_{n-\tau}}{\delta_n}-b\right)z_n, \quad n \geq N. \tag{3.21}$$

From (3.13) and (3.21), we obtain

$$\Delta\left(r_n\Delta z_n\right)+p_n c_{n-k}^{\alpha} z_{n-k}^{\alpha} \leq 0, \quad n \geq N \tag{3.22}$$

where $c_{n-k}^{\alpha}=\left(1-a\frac{\delta_{n-\tau-k}}{\delta_{n-k}}-b\right)^{\alpha}$. From (3.18) and (3.22), we have

$$\begin{aligned}\Delta v_n &\leq -\frac{p_n c_{n-k}^{\alpha} z_{n+1-k}^{\alpha}}{z_{n+1}}-\frac{r_n(\Delta z_n)^2}{z_n z_{n+1}} \\ &\leq -p_n c_{n-k}^{\alpha} z_{n+1-k}^{\alpha-1}-\frac{v_n^2}{r_n}, \quad n \geq N,\end{aligned} \tag{3.23}$$

where we have used $\{z_n\}$ is positive and decreasing. From (3.23), we have

$$\Delta v_n+M_2^{\alpha-1}p_n c_{n-k}^{\alpha}+\frac{v_n^2}{r_n} \leq 0, \quad n \geq N, \tag{3.24}$$

for some constant $M_2=z_{N+1-k}$. Multiplying (3.24) by A_{n+1} and then summing it from N to $n-1$, we have

$$\sum_{s=N}^{n-1}A_{s+1}\Delta v_s+\sum_{s=N}^{n-1}M_2^{\alpha-1}p_s A_{s+1}c_{s-k}^{\alpha}+\sum_{s=N}^{n-1}A_{s+1}\frac{v_s^2}{r_s} \leq 0. \tag{3.25}$$

Using the summation by parts formula in the first term of (3.25), and then rearranging, we obtain

$$A_n v_n-A_N v_N+\sum_{s=N}^{n-1}M_2^{\alpha-1}p_s A_{s+1}c_{s-k}^{\alpha}+\sum_{s=N}^{n-1}\left(\frac{v_s}{r_s}+\frac{v_s^2}{r_s}A_{s+1}\right) \leq 0.$$

Using completing the square in the last term of the above inequality, we obtain

$$\begin{aligned}A_n v_n-A_N v_N+\sum_{s=N}^{n-1}M_2^{\alpha-1}p_s A_{s+1}c_{s-k}^{\alpha}+\sum_{s=N}^{n-1}\frac{A_{s+1}}{r_s}\left(v_s+\frac{1}{2A_{s+1}}\right)^2 \\ -\sum_{s=N}^{n-1}\frac{1}{4r_s A_{s+1}} \leq 0\end{aligned}$$

or

$$A_n v_n \leq A_N v_N-\sum_{s=N}^{n-1}\left(M_2^{\alpha-1}p_s A_{s+1}c_{s-k}^{\alpha}-\frac{1}{4r_s A_{s+1}}\right).$$

Letting $n\to\infty$ in the last inequality and using (3.20), we obtain a contradiction to (3.12). The proof is now completed. □

Theorem 3.5. *Assume that $\alpha \geq 1$, $\left(1 - a - \frac{bR_{n+\sigma-k}}{R_{n-k}}\right) > 0$ for all $n \geq n_0$, and there exists a positive increasing sequence $\{\rho_n\}$ such that for any constant $M > 0$ such that*

$$\sum_{n=n_0}^{\infty} \left[\rho_n p_n \left(1 - a - b\frac{R_{n-\sigma-k}}{R_{n-k}}\right)^{\alpha} - \frac{1}{4\alpha M^{\alpha-1}} \frac{(\Delta\rho_n)^2 r_{n-k}}{\rho_n}\right] = \infty. \tag{3.26}$$

If there exists a positive real sequence $\{\delta_n\}$ such that (3.11) holds and for any constant $D > 0$

$$\sum_{n=n_0}^{\infty} \left[p_n A_{n+1-k}^{\alpha} \left(1 - \frac{a\delta_{n-\tau-k}}{\delta_{n-k}} - b\right)^{\alpha} - \frac{\alpha}{D^{\alpha-1} A_{s-k} r_{s-k}}\right] = \infty \tag{3.27}$$

then every solution of equation (3.9) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.2, we see that Lemma 2.1 holds for all $n \geq N \geq n_0$. Case (I). Assume case (I) of Lemma 2.1 holds. Define

$$w_n = \rho_n \frac{r_n \Delta z_n}{z_{n-k-1}^{\alpha}}, \quad n \geq N. \tag{3.28}$$

Then from (3.28) and (3.5), we obtain

$$\Delta w_n \leq -\rho_n p_n B_{n-k}^{\alpha} + \Delta\rho_n \frac{r_{n+1}\Delta z_{n+1}}{z_{n-k}^{\alpha}} - \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta z_{n-k-1}^{\alpha}}{z_{n-k-1}^{\alpha}}, \quad n \geq N. \tag{3.29}$$

By Mean Value Theorem

$$\Delta z_{n-k-1}^{\alpha} = \alpha t^{\alpha-1} \Delta z_{n-k-1}$$

where $z_{n-k-1} < t < z_{n-k}$. Since $\alpha \geq 1$, we have

$$\Delta z_{n-k-1}^{\alpha} \geq \alpha z_{n-k-1}^{\alpha-1} \Delta z_{n-k-1}. \tag{3.30}$$

Using (3.30) in (3.29) we obtain

$$\Delta w_n \quad \leq \quad -\rho_n p_n B_{n-k}^{\alpha} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \alpha \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{z_{n-k-1}^{\alpha-1} \Delta z_{n-k-1}}{z_{n-k-1}^{\alpha}}, \quad n \geq N. \tag{3.31}$$

Since $\{z_n\}$ is increasing and $\{r_n \Delta z_n\}$ is nonincrasing, we have from (3.31)

$$\Delta w_n \leq -\rho_n p_n B_{n-k}^{\alpha} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \alpha \frac{\rho_n}{\rho_{n+1}^2} \frac{M^{\alpha-1}}{r_{n-k}} w_{n+1}^2$$

where $M = z_{N-k}$. Summing the last inequality from N to $n-1$, and using completing the square we have

$$0 < w_n \leq w_N - \sum_{s=N}^{n-1} \left[\rho_s p_s B_{s-k}^{\alpha} - \frac{1}{4\alpha M^{\alpha-1}} \frac{(\Delta\rho_s)^2 r_{s-k}}{\rho_s}\right].$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (3.26).
Case (II). Assume case (II) of Lemma 2.1 holds. Define

$$v_n = \frac{r_n \Delta z_n}{z_{n-k}^{\alpha}}, \quad n \geq N. \tag{3.32}$$

Then $v_n < 0$ for $n \geq N$. From Lemma 2.4, we have

$$A_{n-k}\frac{r_n\Delta z_n}{z_{n-k}} \geq -1, \quad n \geq N.$$

Then

$$-\frac{r_n\Delta z_n(-r_n\Delta z_n)^{\alpha-1}}{z_{n-k}^{\alpha}}A_{n-k}^{\alpha} \leq 1.$$

So, by $\Delta(-r_n\Delta z_n) > 0$ and (3.32), we have

$$-\frac{1}{D^{\alpha-1}} \leq v_n A_{n-k}^{\alpha} \leq 0, \quad n \geq N, \tag{3.33}$$

where $D = -r_N\Delta z_N$. From (3.32), we have

$$\Delta v_n = \frac{\Delta(r_n\Delta z_n)}{z_{n-k+1}^{\alpha}} - \frac{r_n\Delta z_n}{z_{n-k}^{\alpha}z_{n-k+1}^{\alpha}}\Delta z_{n-k}^{\alpha}.$$

By Mean Value Theorem

$$\Delta z_{n-k}^{\alpha} = \alpha t^{\alpha-1}\Delta z_{n-k}$$

where $z_{n-k+1} < t < z_{n-k}$. Since $\alpha \geq 1$ and $\Delta z_{n-k} < 0$, we have

$$\Delta z_{n-k}^{\alpha} \leq \alpha z_{n-k+1}^{\alpha-1}\Delta z_{n-k}.$$

Therefore

$$\Delta v_n \leq -\frac{p_n x_{n-k+1}^{\alpha}}{z_{n+1-k}^{\alpha}} - \frac{\alpha r_n\Delta z_n}{z_{n-k}^{\alpha}z_{n-k+1}}\Delta z_{n-k}. \tag{3.34}$$

From (3.21) and (3.34), we obtain

$$\Delta v_n + p_n c_{n-k}^{\alpha} \leq 0, \quad n \geq N. \tag{3.35}$$

Multiplying (3.35) by A_{n+1}^{α} and then summing it from N to $n-1$, we have

$$\sum_{s=N}^{n-1} A_{s+1-k}^{\alpha}\Delta v_s + \sum_{s=N}^{n-1} A_{s+1-k}^{\alpha}p_s c_{s-k}^{\alpha} \leq 0. \tag{3.36}$$

Summation by parts formula yields

$$\sum_{s=N}^{n-1} A_{s+1-k}^{\alpha}\Delta v_s = A_{n-k}^{\alpha}v_n - A_{N-k}^{\alpha}v_N - \sum_{s=N}^{n-1} v_s\Delta A_{s-k}^{\alpha}.$$

By Mean Value Theorem, we obtain

$$\Delta A_{s-k}^{\alpha} \geq -\frac{\alpha A_{s-k}^{\alpha-1}}{r_{s-k}}.$$

Since $v_n < 0$, we have

$$\sum_{s=N}^{n-1} A_{s+1-k}^{\alpha}\Delta v_s \geq A_{n-k}^{\alpha}v_n - A_{N-k}^{\alpha}v_N + \sum_{s=N}^{\alpha-1}\frac{\alpha v_s A_{s-k}^{\alpha-1}}{r_{s-k}}. \tag{3.37}$$

Using (3.37) in (3.36), we obtain

$$A_{n-k}^{\alpha}v_n - A_{N-k}^{\alpha}v_N + \sum_{s=N}^{n-1}\frac{\alpha v_s A_{s-k}^{\alpha-1}}{r_{s-k}} + \sum_{s=N}^{n-1} p_s A_{s+1-k}^{\alpha}c_{s-k}^{\alpha} \leq 0. \tag{3.38}$$

Therefore from (3.33) and (3.38), we obtain

$$-\frac{1}{D^{\alpha-1}} \leq A^{\alpha}_{n-k}v_n \leq A^{\alpha}_{N-k}v_N - \sum_{s=N}^{n-1}\left[p_s A^{\alpha}_{s+1-k}c^{\alpha}_{s-k} - \frac{\alpha}{D^{\alpha-1}A_{s-k}r_{s-k}}\right],$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (3.27). This ends the proof. □

Theorem 3.6. *Assume that $\alpha \geq 1$, $\left(1 - a - \frac{bR_{n+\sigma-k}}{R_{n-k}}\right) > 0$ for all $n \geq n_0$, and there exists a positive nondecreasing sequence $\{\rho_n\}$ such that for any constant $M > 0$, (3.26) holds. If there exists a positive real sequence $\{\delta_n\}$ such that (3.11) holds, and*

$$\sum_{n=n_0}^{\infty}\frac{1}{r_n}\sum_{s=n_0}^{n-1}p_s A^{\alpha}_{s-k+1}\left(1 - \frac{a\delta_{n-\tau-k}}{\delta_{n-k}} - b\right)^{\alpha} = \infty \tag{3.39}$$

then every solution of equation (3.9) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.3, we see that Lemma 2.1 holds and Case (I) is eliminated by the condition (3.26).

Case (II). Proceeding as in the proof of Theorem 3.2 (Case (II)), we have

$$z_{n-k} \geq -r_n\Delta z_n A_{n-k} \geq -r_N\Delta z_N A_{n-k} = dA_{n-k}$$

where $d = -r_N\Delta z_N$. From (3.22) we have

$$\Delta(-r_n\Delta z_n) \geq d^{\alpha}p_n c^{\alpha}_{n-k}A^{\alpha}_{n+1-k}, \quad n \geq N.$$

Summing the last inequality from N to $n-1$, we obtain

$$-r_n\Delta z_n \geq -r_N\Delta z_N + d^{\alpha}\sum_{s=N}^{n-1}p_s c^{\alpha}_{s-k}A^{\alpha}_{s+1-k}.$$

Dividing the last inequality by r_n and then summing from N to $n-1$, we have

$$z_N \geq z_N - z_n \geq d^{\alpha}\sum_{s=N}^{n-1}\frac{1}{r_s}\sum_{t=N}^{s-1}p_t c^{\alpha}_{t-k}A^{\alpha}_{t+1-k}.$$

Letting $n \to \infty$ in the last inequality, we obtain

$$\sum_{n=N}^{\infty}\frac{1}{r_n}\sum_{s=N}^{n-1}p_s c^{\alpha}_{s-k}A^{\alpha}_{s+1-k} \leq z_N$$

a contradiction to (3.39). This ends the proof. □

4 Examples

In this section, we present some examples to illustrate the main results.

Example 4.1. *Consider the neutral difference equation*

$$\Delta\left(2^{n+1}\Delta\left(x_n + \frac{1}{2}x_{n-2} + \frac{1}{4}x_{n+1}\right)\right) + 22(2^n)x_{n-3} + 2^n x_{n+2} = 0, \quad n \geq 1. \tag{4.1}$$

Here, $r_n = 2^{n+1}$, $\alpha = \beta = 1$, $a = 1/2$, $b = 1/4$, $p_n = 22(2^n)$, $q_n = 2^n$, $\tau = 2$, $\sigma = 2$, $k = 3$, and $m = 2$. Then $R_n = \frac{1}{2} - \frac{1}{2^n}$ and $A_n = \frac{1}{2^n}$. It is easy to verify that all condition of Corollary 3.1 are satisfied, and hence every solution of equation (3.41) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such solution of equation (4.1).

Example 4.2. *Consider the neutral difference equation*

$$\Delta\left(2^{n+1}\Delta\left(x_n+\frac{3}{4}x_{n-1}+\frac{1}{2}x_{n+2}\right)\right)+exp(e^{4n})x_{n-2}^3+2^nx_{n+2}=0,\quad n\geq 1. \tag{4.2}$$

Here, $r_n=2^{n+1}$, $\alpha=3$, $\beta=1$, $a=\frac{3}{4}$, $b=\frac{1}{2}$, $p_n=exp(e^{4n})$, $q_n=2^n$, $\tau=1$, $\sigma=2$, $k=2$, *and* $m=2$. *Then* $R_n=\frac{1}{2}-\frac{1}{2^n}$ *and* $A_n=\frac{1}{2^n}$. *It is easy to verify that all conditions of Corollary 3.2 are satisfied, and hence every solution of equation (4.2) is oscillatory.*

Example 4.3. *Consider the neutral difference equation*

$$\Delta\left(2^{n+1}\Delta\left(x_n+\frac{1}{16}x_{n-2}+\frac{1}{8}x_{n+1}\right)\right)+45(2^{n-2})x_{n-1}^{1/3}=0,\quad n\geq 1. \tag{4.3}$$

Here, $r_n=2^{n+1}$, $\alpha=\frac{1}{3}$, $a=\frac{1}{16}$, $b=\frac{1}{8}$, $p_n=45(2^{n-2})$, $\tau=2$, $\sigma=1$, *and* $k=2$. *Then* $R_n=\frac{1}{2}-\frac{1}{2^n}$ *and* $A_n=\frac{1}{2^n}$. *By taking* $\rho_n=1$ *and* $\delta=\frac{1}{2^n}$, *it is easy to see that all conditions of Theorem 3.4 are satisfied, and hence every solution of equation (4.3) is oscillatory. In fact* $\{x_n\}=\left\{(-1)^{3n}\right\}$ *is one such solution of equation (4.3).*

5 Conclusion

We conclude this paper with the following remark:

Remark 5.1. The results given in [2], [3], [4], [5] and [8] can not be applicable to the equations (4.1) to (4.3) since these equations contain advanced terms. Also the results in [9], [10], [11], [12], [14], [15] and [16] can not be applicable to the equations (4.1) to (4.3) since the condition (C1) is not satisfied. Further from the theorems in [17] we conclude only that all solutions of equations (4.1) to (4.3) are either oscillatory or tend to zero monotonically. But the theorems obtained here ensure that all solutions of equations (4.1) to (4.3) are oscillatory.Therefore our results improve and further complement to some of the results in [2], [3], [4], [5], [17] and [8].

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Competing Interests

The authors declare that no competing interests exist.

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