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# Oscillatory Mixed Differential Systems

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**Abstract.** In this work are obtained some criteria which guarantee the oscillatory behavior of the differential system of mixed type $x'(t) = \int_{-1}^{0} d[\nu(\theta)]x(t - r(\theta)) + \int_{-1}^{0} d[\eta(\theta)]x(t + \tau(\theta))$, where $x(t) \in \boldsymbol{R}^n$, $r(\theta)$ and $\tau(\theta)$ are real nonnegative continuous functions on $[-1, 0]$, $\nu(\theta)$ and $\eta(\theta)$ are real $n$-by-$n$ matrix valued function of bounded variation on $[-1, 0]$.

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## 1. Introduction

In this note is considered the differential difference system of mixed type

$$x'(t) = \sum_{i=1}^{p} A_i x(t - r_i) + \sum_{j=1}^{q} B_j x(t + \tau_j), \tag{1}$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T \in \boldsymbol{R}^n$, the $A_i$ and $B_j$ are $n$-by-$n$ real matrices and the $r_i$ and $\tau_j$ are positive real numbers. Systems of this kind can arise in the study on travelling waves in domains with nonlocal interactions initiated in [1], [2], and more recently developed in [3], [4].

Equation (1) can be looked in the more general framework of the mixed functional differential system

$$x'(t) = \int_{-1}^{0} d[\nu(\theta)]x(t - r(\theta)) + \int_{-1}^{0} d[\eta(\theta)]x(t + \tau(\theta)), \tag{2}$$

where $r(\theta)$ and $\tau(\theta)$ are real nonnegative continuous functions on $[-1, 0]$, $\nu(\theta)$ and $\eta(\theta)$ are real $n$-by-$n$ matrix valued functions of bounded variation on $[-1, 0]$. Some interest on these equations can be noticed in the area of economic research, as it can be seen in [5], [6] and references therein.

The aim of this work is to complement the study made in [7] on the oscillatory behavior of systems (1) and (2). For that purpose we notice

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that system (1) corresponds to have in (2), $v$ and $\eta$ as step functions of the form

$$v(\theta) = \sum_{i=1}^{p} H(\theta - \theta_i)A_i, \qquad \eta(\theta) = \sum_{j=1}^{q} H(\theta - \xi_j)B_j, \tag{3}$$

where $H$ denotes the Heaviside function, $-1 < \theta_1 < \cdots < \theta_p < 0$ and $-1 < \xi_1 < \cdots < \xi_q < 0$, and $r(\theta)$ and $\tau(\theta)$ are continuous and positive functions on $[-1, 0]$ such that $r(\theta_i) = r_i$ and $\tau(\xi_j) = \tau_j$, for $i = 1, \dots, p$ and $j = 1, \dots, q$.

Considering the value $\|r\| = \max\{r(\theta) : -1 \le \theta \le 0\}$, by a solution of (2) we mean a continuous function $x : [-\|r\|, +\infty[ \to \boldsymbol{R}^n$, which is differentiable on $]0, +\infty[$ and satisfies (2) for every $t > 0$.

Taking the space $C = C([-\|r\|, \|\tau\|]; \boldsymbol{R}^n)$, let $\phi \in C$. Considering the initial condition $x(\theta) = \phi(\theta)$ for $\theta \in [-\|r\|, \|\tau\|]$, the initial value problem associated to (2) may be ill-posed (see [8]). This leads to some questions concerning the admissibility of the function spaces for abstract Cauchy problems of functional differential equations. On this subject we call the attention of the reader to [9] and [10].

A set $\mathscr{K} \subset \boldsymbol{R}^n$ is called a cone if for each $u, v \in \mathscr{K}$ and $a \ge 0$, $b \ge 0$, one has $au + bv \in \mathscr{K}$, and $-u \notin \mathscr{K}$ whenever $u \in \mathscr{K} \setminus \{0\}$. Then following [7], a solution $x(t)$ of (2) will be said *nonoscillatory* if there exists a $T \ge 0$ and a closed cone $\mathscr{K} \subset \boldsymbol{R}^n$ such that $x(t) \in \mathscr{K} \setminus \{0\}$ for all $t \ge T$; otherwise $x(t)$ is called *oscillatory*. This definition of oscillatory solution seems to be a middle situation between the concepts of oscillatory componentwise and weakly oscillatory solutions introduced in [11].

Under a functional perspective, a slightly stronger notion of oscillation can also be considered. Taking a solution $x : [-\|r\|, +\infty[ \to \boldsymbol{R}^n$ of (2) satisfying the initial condition $x(\theta) = \phi(\theta)$ for $\theta \in [-\|r\|, \|\tau\|]$, through the relation $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\|r\|, \|\tau\|]$ and $t \ge 0$, one can induce a semidynamical system on the space $C$. Then following [12] and [13] one can say that a nontrivial solution $x(t)$ is *strongly nonoscillatory* if there exists $\zeta$, a real 1-by-$n$ matrix valued function of bounded variation on $[-\|r\|, \|\tau\|]$, and $T > 0$ such that

$$\int_{-\|r\|}^{\|\tau\|} d[\zeta(\theta)]x(t + \theta) \ge 0,$$

for all $t \ge T$. Otherwise that solution is said *strongly oscillatory*. Under a geometric point of view, one can say that $x$ is strongly nonoscillatory if and only if $x_t$ is contained in some closed half-space of $C$, for every $t \ge T$.

For the delay case,

$$x'(t) = \int_{-1}^{0} d[v(\theta)]x(t - r(\theta)), \tag{4}$$

corresponding to have $\tau = 0$ and $\eta = 0$ in (2), let's consider the characteristic equation

$$\det\left(\lambda I - \int_{-1}^{0} \exp(\lambda r(\theta)) d[\nu(\theta)]\right) = 0,$$

where $I$ denotes the $n$-by-$n$ identity matrix. Then (4) is *oscillatory*, that is, has all solutions oscillatory, if and only if the characteristic equation has no real roots (see [11] and [14]). For strong oscillations this property is not valid as is shown through an example in [12]. The same holds in the regard of the mixed system

$$x'(t) = \int_{-1}^{1} d[\psi(s)] x(t+s) \tag{5}$$

with $\psi : [-1, 1] \to \boldsymbol{R}^{n \times n}$ of bounded variation. In fact, in [7] is also given an example where the characteristic equation

$$\det\left(\lambda I - \int_{-1}^{1} \exp(\lambda s) d[\psi(s)]\right) = 0$$

has no real roots and (5) has a nonoscillatory solution.

Contrarily to the delay case, for mixed type systems the real part of the spectral set $\{\mathrm{Re}\,\lambda : \det \Delta(\lambda) = 0\}$ is not in general bounded above and the solutions may not be exponentially bounded (see [7], [8] and [14]). These difficulties are overcame in [7] by assuming that:

There exist a nonsingular matrix $A \in \boldsymbol{R}^{n \times n}$ and a nondecreasing function $\gamma : [-1, 1] \to \boldsymbol{R}$ such that $\gamma(s) < \gamma(1)$ for all $s \in [-1, 1)$, and

$$\frac{|\psi A - \gamma I|(1) - |\psi A - \gamma I|(s)}{\gamma(1) - \gamma(s)} \to 0 \qquad \text{as } s \to 1^-, \tag{H}$$

where by $|\phi|(s)$ is meant the total variation on $[-1, s]$ of any $n$-by-$n$ matrix valued function $\phi$ of bounded variation on $[-1, 1]$.

The assumption (H) holds, for example, either if

$$\psi(1) - \psi(1^-) \text{ is a nonsingular matrix,} \tag{H1}$$

or if there are a nonsingular matrix $B \in \boldsymbol{R}^{n \times n}$, $s_0 \in [-1, 1)$, and a monotone function $\chi : [s_0, 1] \to \boldsymbol{R}$ so that

$$\begin{aligned} &\psi(s) B = \chi(s) I \quad \text{for all } s \in [s_0, 1] \qquad \text{and} \\ &\chi(s) \neq \chi(1) \quad \text{for all } s \in [s_0, 1). \end{aligned} \tag{H2}$$

The condition (H1) means that the operator $L : C([-1,1], \boldsymbol{R}^n)$ given by

$$\phi \mapsto \int_{-1}^{1} d[\psi(s)]\phi(s)$$

is atomic at 1 in the sense of [15], while through (H2), (H) may be satisfied in cases where $L$ is nonatomic at 1.

Comparatively to system (5), there is no gain in generality by considering the system (2). However, as we will see, there is some convenience in separating advances and delays. For a matter of simplicity and brevity of exposition, we will assume that the delay function $r(\theta)$ and the advance function $\tau(\theta)$ are increasing on $[-1,0]$. However the methods we will use in the sections 1 and 2 can be applied, more generally, to delays and advances both monotonous as well as to delays and advances exhibiting some oscillations. On this subject, if $\phi$ is a real function on $[-1,0]$, whenever $\phi$ is said to be increasing or decreasing on a interval $J \subset [-1,0]$, we exclude the possibility of $\phi$ be constant on $J$.

Therefore in order that be verified a similar condition to (H1) we will suppose hereafter that the advance function $\tau(\theta)$ is such that $\tau(\theta) < \tau(0)$ for every $\theta \in [-1,0[$, and that $\eta(0) - \eta(0^-)$ is a nonsingular matrix. In the concerning of system (1) we will assume the delays and advances ordered as $r_1 < \cdots < r_p$, $\tau_1 < \cdots < \tau_q$, and that the matrix $B_q$ corresponding to the largest advance is non singular.

Letting then

$$\Delta(\lambda) = \lambda \boldsymbol{I} - \left( \int_{-1}^{0} \exp(-\lambda r(\theta)) d[\nu(\theta)] + \int_{-1}^{0} \exp(\lambda \tau(\theta)) d[\eta(\theta)] \right),$$

supposing that $\det \Delta(\lambda) = 0$, for some $\lambda \in \boldsymbol{R}$, one easily sees that then for some $u \in \boldsymbol{R}^n \backslash \{0\}$, the eigenvector $\exp(\lambda\theta)u$ is a nonoscillatory solution of the system (2). Moreover according to [7] one can summarize the main characteristics of that system in the following lemma.

**Lemma 1.** i) $\{\lambda \in \boldsymbol{R} : \det \Delta(\lambda) = 0\}$ *is bounded above.*

ii) *Any nonoscillatory solution of* (2) *is exponentially bounded.*

iii) *If* (2) *has a nonoscillatory solution then*

$$\det \Delta(\lambda) = 0$$

*has a real root.*

iv) (2) *is oscillatory if and only if*

$$\det \Delta(\lambda) = 0$$

*has no real roots.*

Therefore defining

$$M(\lambda) = \int_{-1}^{0} \exp(-\lambda r(\theta)) d[v(\theta)] + \int_{-1}^{0} \exp(\lambda \tau(\theta)) d[\eta(\theta)],$$

the system (2) is oscillatory if and only

$$\lambda \notin \sigma(M(\lambda)), \qquad \text{for every real } \lambda. \tag{6}$$

Nonoscillatory solutions will exist, whenever

$$\det[\lambda I - M(\lambda)] = 0 \tag{7}$$

has at least a real root.

As in many papers on the oscillatory theory of delay equations, matrix measures also constitute here a relevant tool for the analysis of the oscillatory behavior of (2) and (4). For the definition and its main properties we will follow [16] and [17].

Notice that since for any matrix measure $\mu$ one has

$$\operatorname{Re} \sigma(M(\lambda)) \subset [-\mu(-M(\lambda)), \mu(M(\lambda))],$$

if either

$$\mu(M(\lambda)) < \lambda, \qquad \text{for every real } \lambda, \tag{8}$$

or

$$\lambda < -\mu(-M(\lambda)), \qquad \text{for every real } \lambda, \tag{9}$$

then (6) is satisfied.

## 2. Continuous delays and advances

For a given function $\phi$ of bounded variation on $[-1,0]$, we will associate the functions $\phi_0$ and $\phi_1$, also of bounded variation, defined, respectively, by

$$\phi_0(\theta) = \phi(0) - \phi(\theta), \quad \phi_1(\theta) = \phi(\theta) - \phi(-1) \qquad (\theta \in [-1,0]).$$

By $\Delta\phi$ we will mean the difference $\phi(0) - \phi(-1) = \phi_0(-1) = \phi_1(0)$.

**Theorem 2.** *Let $\mu$ be a matrix measure.*

a) *If*

$$\mu \circ v_1 \text{ and } \mu \circ \eta_1 \text{ are decreasing}, \tag{10}$$

$$\mu \circ v_0 \text{ and } \mu \circ \eta_0 \text{ are increasing}, \tag{11}$$

$$\int_{-1}^{0} r(\theta) d(\mu \circ v_0)(\theta) > e^{-1}, \tag{12}$$

*then* (2) *is oscillatory independently of the advances.*

b) *If*

$$\mu \circ (-\nu_1) \text{ and } \mu \circ (-\eta_1) \text{ are decreasing,} \tag{13}$$

$$\mu \circ (-\nu_0) \text{ and } \mu \circ (-\eta_0) \text{ are increasing,} \tag{14}$$

$$\int_{-1}^{0} \tau(\theta) d(\mu \circ (-\eta_0))(\theta) > e^{-1}. \tag{15}$$

*then* (2) *is oscillatory independently of the delays.*

*Proof.* a) We first notice that by the subadditivity property of a matrix measure we have

$$\mu(M(\lambda)) \leq \mu\left(\int_{-1}^{0} \exp(-\lambda r(\theta)) d[\nu(\theta)]\right) + \mu\left(\int_{-1}^{0} \exp(\lambda \tau(\theta)) d[\eta(\theta)]\right). \tag{16}$$

Therefore $\mu(M(0)) \leq \mu(\Delta\nu) + \mu(\Delta\eta)$. As by (10) and (11)

$$\mu(\Delta\nu) = (\mu \circ \nu_1)(0) < (\mu \circ \nu_1)(-1) = 0$$

and

$$\mu(\Delta\eta) = (\mu \circ \eta_1)(0) < (\mu \circ \eta_1)(-1) = 0,$$

then $\mu(M(0)) < 0$ and (8) is verified for $\lambda = 0$.

On the other hand, by the properties of the matrix measures obtained in [17, Lemma 2.2, (i) and (ii)], one has

$$\mu\left(\int_{-1}^{0} \exp(-\lambda r(\theta)) d[\nu(\theta)]\right) \leq \begin{cases} \int_{-1}^{0} \exp(-\lambda r(\theta)) d(\mu \circ \nu_1)(\theta), & \text{if } \lambda > 0, \\ -\int_{-1}^{0} \exp(-\lambda r(\theta)) d(\mu \circ \nu_0)(\theta), & \text{if } \lambda < 0, \end{cases} \tag{17}$$

and

$$\mu\left(\int_{-1}^{0} \exp(\lambda \tau(\theta)) d[\eta(\theta)]\right) \leq \begin{cases} -\int_{-1}^{0} \exp(\lambda \tau(\theta)) d(\mu \circ \eta_0)(\theta), & \text{if } \lambda > 0, \\ \int_{-1}^{0} \exp(\lambda \tau(\theta)) d(\mu \circ \eta_1)(\theta), & \text{if } \lambda < 0. \end{cases} \tag{18}$$

Then letting $\lambda > 0$, by (16), (17) and (18) we have

$$\begin{aligned} \mu(M(\lambda)) \leq & \int_{-1}^{0} \exp(-\lambda r(\theta)) d(\mu \circ \nu_1)(\theta) \\ & - \int_{-1}^{0} \exp(\lambda \tau(\theta)) d(\mu \circ \eta_0)(\theta). \end{aligned} \tag{19}$$

So, $\mu(M(\lambda)) \leq 0 < \lambda$ for every $\lambda > 0$.

Let now $\lambda < 0$. Analogously one obtains

$$\mu(M(\lambda)) \le -\int_{-1}^{0} \exp(-\lambda r(\theta)) d(\mu \circ v_0)(\theta) \tag{20}$$
$$+ \int_{-1}^{0} \exp(\lambda \tau(\theta)) d(\mu \circ \eta_1)(\theta).$$

Therefore, for every $\lambda < 0$ one has

$$\mu(M(\lambda)) \le -\int_{-1}^{0} \exp(-\lambda r(\theta)) d(\mu \circ v_0)(\theta).$$

Then noticing that $\exp u \ge eu$ for every $u$, (12) implies that

$$\mu(M(\lambda)) \le \lambda e \int_{-1}^{0} r(\theta) d(\mu \circ v_0)(\theta) < \lambda,$$

for every $\lambda < 0$.

Thus (8) is satisfied.

b) Similar arguments will enable us to conclude (9).

Now by the subadditivity property of a matrix measure we have

$$\mu(-M(\lambda)) \le \mu\left(\int_{-1}^{0} (-\exp(-\lambda r(\theta))) d[v(\theta)]\right) \tag{21}$$
$$+ \mu\left(\int_{-1}^{0} (-\exp(\lambda \tau(\theta))) d[\eta(\theta)]\right).$$

Hence for $\lambda = 0$, (13) and (14) imply that

$$\mu(-M(0)) \le \mu(-\Delta v) + \mu(-\Delta \eta) = (\mu \circ (-v_1))(0) + (\mu \circ (-\eta_1))(0) < 0,$$

and then $0 < -\mu(-M(0))$.

By the properties of the matrix measures expressed in [17, Lemma 2.2, (iii) and (iv)], the following inequalities hold:

$$\mu\left(\int_{-1}^{0} (-\exp(-\lambda r(\theta))) d[v(\theta)]\right) \tag{22}$$
$$\le \begin{cases} \int_{-1}^{0} \exp(-\lambda r(\theta)) d(\mu \circ (-v_1))(\theta), & \text{if } \lambda > 0, \\ -\int_{-1}^{0} \exp(-\lambda r(\theta)) d(\mu \circ (-v_0))(\theta), & \text{if } \lambda < 0, \end{cases}$$

and

$$\mu\left(\int_{-1}^{0} (-\exp(\lambda \tau(\theta))) d[\eta(\theta)]\right) \tag{23}$$
$$\le \begin{cases} -\int_{-1}^{0} \exp(\lambda \tau(\theta)) d(\mu \circ (-\eta_0))(\theta), & \text{if } \lambda > 0, \\ \int_{-1}^{0} \exp(\lambda \tau(\theta)) d(\mu \circ (-\eta_1))(\theta), & \text{if } \lambda < 0. \end{cases}$$

Thus for $\lambda < 0$ one has by (21), (22) and (23)

$$\mu(-M(\lambda)) \leq -\int_{-1}^{0} \exp(-\lambda r(\theta)) d(\mu \circ (-v_0))(\theta) + \int_{-1}^{0} \exp(\lambda \tau(\theta)) d(\mu \circ (-\eta_1))(\theta). \tag{24}$$

So, by (13) and (14) it is $\mu(-M(\lambda)) \leq 0 < -\lambda$ for every $\lambda < 0$.

For $\lambda > 0$, one obtains analogously

$$\mu(-M(\lambda)) \leq \int_{-1}^{0} \exp(-\lambda r(\theta)) d(\mu \circ (-v_1))(\theta) - \int_{-1}^{0} \exp(\lambda \tau(\theta)) d(\mu \circ (-\eta_0))(\theta). \tag{25}$$

Therefore, for every $\lambda > 0$ one has as before through (15)

$$\begin{aligned} \mu(-M(\lambda)) &\leq -\int_{-1}^{0} \exp(\lambda \tau(\theta)) d(\mu \circ (-\eta_0))(\theta) \\ &\leq -\lambda e \int_{-1}^{0} \tau(\theta) d(\mu \circ (-\eta_0))(\theta) < -\lambda. \end{aligned}$$

Thus (9) is satisfied. ■

*Remark* 3. Adding to parts a) and b) of Theorem 2, respectively, the conditions

$$\int_{-1}^{0} r(\theta) d(\mu \circ v_1)(\theta) < -e^{-1}, \tag{26}$$

and

$$\int_{-1}^{0} r(\theta) d(\mu \circ (-\eta_1))(\theta) < -e^{-1}, \tag{27}$$

identical arguments lead us to the same conclusions for monotonous delays and advances. In that way one obtains an extension of [18, Theorem 1] for monotonic delays and advances. With respect to the results obtained in this work, for systems we are very much more restricted by the properties of the matrix measures.

*Remark* 4. We notice that since

$$\int_{-1}^{0} r(\theta) d(\mu \circ v_0)(\theta) \geq -\mu(\Delta v) \min_{\theta \in [-1,0]} r(\theta),$$

(12) is satisfied if

$$\mu(\Delta v) \min_{\theta\in[-1,0]} r(\theta) < -e^{-1}.$$

Similarly, from

$$\int_{-1}^{0} \tau(\theta) d(\mu \circ (-\eta_0))(\theta) \geq -\mu(-\Delta\eta) \min_{\theta\in[-1,0]} \tau(\theta),$$

one has (15) verified if

$$\mu(-\Delta\eta) \min_{\theta\in[-1,0]} \tau(\theta) < -e^{-1}.$$

In the regard of the system (1), for a given matrix measure $\mu$, following [10] we consider the measures $\alpha$ and $\beta$ relatively to a given ordered family of matrices $(C_1, \dots, C_m)$:

$$\alpha(C_1) = \mu(C_1), \qquad \alpha(C_j) = \mu\left(\sum_{k=1}^{j} C_k\right) - \mu\left(\sum_{k=1}^{j-1} C_k\right) \quad \text{for } j = 2, \dots, p,$$

$$\beta(C_p) = \mu(C_p), \qquad \beta(C_j) = \mu\left(\sum_{k=j}^{p} C_k\right) - \mu\left(\sum_{k=j+1}^{p} C_k\right) \quad \text{for } j = 1, \dots, p-1.$$

Observe that by the subadditivity of a matrix measure one easily sees that

$$\alpha(C_j) \leq \mu(C_j), \qquad \beta(C_j) \leq \mu(C_j), \quad j = 1, \dots, m. \tag{28}$$

Moreover if, as in (3), for $\theta \in [-1,0]$, $\phi(\theta) = \sum_{k=1}^{m} H(\theta - \theta_i) C_i$, with $-1 < \theta_1 < \cdots < \theta_m < 0$, then

$$(\mu \circ \phi_0)(\theta) = \mu\left(\sum_{i=1}^{m} (1 - H(\theta - \theta_i)) C_i\right)$$

and

$$(\mu \circ \phi_1)(\theta) = \mu\left(\sum_{i=1}^{m} H(\theta - \theta_i) C_i\right)$$

are step functions with jumps at $\theta_k$ equal, respectively, to

$$-\beta(C_k) = \mu\left(\sum_{i=k+1}^{m} C_i\right) - \mu\left(\sum_{i=k}^{m} C_i\right), \qquad \alpha(C_k) = \mu\left(\sum_{i=1}^{k} C_i\right) - \mu\left(\sum_{i=1}^{k-1} C_i\right).$$

In the same way, by considering the sequence $(-C_1, \dots, -C_m)$ one has

$$(\mu \circ (-\phi_0))(\theta) = \mu\left(\sum_{i=1}^{m}(1 - H(\theta - \theta_i))(-C_i)\right)$$

and

$$(\mu \circ (-\phi_1))(\theta) = \mu\left(\sum_{i=1}^{m} H(\theta - \theta_i)(-C_i)\right),$$

which are step functions with jumps at $\theta_k$ equal to, respectively,

$$-\beta(-C_k) = \mu\left(-\sum_{i=k+1}^{m} C_i\right) - \mu\left(-\sum_{i=k}^{m} C_i\right),$$

$$\alpha(-C_k) = \mu\left(-\sum_{i=1}^{k} C_i\right) - \mu\left(-\sum_{i=1}^{k-1} C_i\right).$$

Therefore if for every $k = 1, \dots, m$ one has $\beta(C_k) < 0$, then $\mu \circ \phi_0$ is increasing and if for every $k = 1, \dots, m$ it is $\alpha(C_k) < 0$ then $\mu \circ \phi_1$ is decreasing.

Thus the following corollary holds.

**Corollary 5.** a) *Let* $\alpha(A_i) < 0$, $\beta(A_i) < 0$, *for every* $i = 1, \dots, p$, *and* $\alpha(B_j) < 0$, $\beta(B_j) < 0$, *for every* $j = 1, \dots, q$. *If*

$$\sum_{i=1}^{p} r_i \beta(A_i) < -e^{-1} \tag{29}$$

*then* (1) *is oscillatory, independently of the advances* $\tau_1 < \cdots < \tau_q$.

b) *Let* $\alpha(-A_i) < 0$, $\beta(-A_i) < 0$, *for every* $i = 1, \dots, p$, *and* $\alpha(-B_j) < 0$, $\beta(-B_j) < 0$, *for every* $j = 1, \dots, q$. *If*

$$\sum_{j=1}^{q} \tau_j \beta(-B_j) < -e^{-1} \tag{30}$$

*then* (1) *is oscillatory, independently of the delays* $r_1 < \cdots < r_q$.

*Proof.* a) By the comments above, (10) and (11) are satisfied. On the other hand (12) is in this case equivalent to (29). The part b) follows analogously. ∎

As a consequence of this corollary, by (28) we conclude the following statements.

**Corollary 6.** a) *Let* $\mu(A_i) < 0$ *for every* $i = 1, \dots, p$, *and* $\mu(B_j) < 0$ *for every* $j = 1, \dots, q$. *If*

$$\sum_{i=1}^{p} r_i \mu(A_i) < -e^{-1}, \tag{31}$$

*then* (1) *is oscillatory, independently of the advances* $\tau_1 < \cdots < \tau_q$.

b) *Let* $\mu(-A_i) < 0$ *for every* $i = 1, \dots, p$, *and* $\mu(-B_j) < 0$ *for every* $j = 1, \dots, q$. *If*

$$\sum_{j=1}^{q} \tau_j \mu(-B_j) < -e^{-1}, \tag{32}$$

*then* (1) *is oscillatory, independently of the delays* $r_1 < \cdots < r_q$.

*Remark* 7. Assuming that $r_1 < \cdots < r_p$, $\tau_1 < \cdots < \tau_q$, by the Remark 4, the conditions (31) and (32) in the Corollary 6 can be replaced, respectively, by

$$r_1 \mu\left(\sum_{i=1}^{p} A_i\right) < -e^{-1} \qquad \text{and} \qquad \tau_1 \mu\left(\sum_{j=1}^{q} B_j\right) < -e^{-1}.$$

We illustrate the Corollary 5 through the following example using the well-known matrix measure $\mu_1$ of a matrix $C = [c_{jk}] \in \boldsymbol{M}_n(\boldsymbol{R})$ given by

$$\mu_1(C) = \max_{1 \leqslant k \leqslant n} \left\{ c_{kk} + \sum_{j \neq k} |c_{jk}| \right\}.$$

*Example* 8. Let the system (1) with $r_1 = 1/8$, $r_2 = 1/4$, $r_3 = 1/2$ and

$$A_1 = \begin{bmatrix} -5 & 2 \\ 1 & -4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} -6 & 1 \\ 2 & -7 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -7 & 1 \\ -1 & -4 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix}.$$

Using $\mu_1$, we have

$$\alpha(A_1) = \mu_1(A_1) = \max\{-4, -2\} = -2,$$

$$\alpha(A_2) = \mu_1(A_1 + A_2) - \mu_1(A_1) = \max\{-3, -5\} + 2 = -1,$$

$$\alpha(A_3) = \mu_1(A_1 + A_2 + A_3) - \mu_1(A_1 + A_2) = \max\{-9, -11\} + 1 = -8,$$

$$\beta(A_3) = \mu_1(A_3) = \max\{-4, -6\} = -4,$$

$$\beta(A_2) = \mu_1(A_2 + A_3) - \mu_1(A_3) = \max\{-5, -7\} + 4 = -1,$$

$$\beta(A_1) = \mu_1(A_1 + A_2 + A_3) - \mu_1(A_2 + A_3) = -9 + 1 = -8,$$

and

$$\alpha(B_1) = \mu_1(B_1) = \max\{-6, -3\} = -3,$$
$$\alpha(B_2) = \mu_1(B_1 + B_2) - \mu_1(B_1) = \max\{-8, -6\} + 3 = -3,$$
$$\beta(B_2) = \mu_1(B_2) = \max\{-2, -1\} = -1,$$
$$\beta(B_1) = \mu_1(B_1 + B_2) - \mu_1(B_2) = \max\{-8, -6\} + 1 = -5.$$

Since

$$\sum_{i=1}^{3} r_i \beta(A_i) = -1 - \frac{1}{4} - 2 = -\frac{13}{4} < -e^{-1},$$

by Corollary 5 a), the correspondent (1) is oscillatory independently of the advances $\tau_1 < \tau_2$.

## 3. Differentiable delays and advances

In this section we will assume further that $r(\theta)$ and $\tau(\theta)$ are differentiable functions on $[-1, 0]$. In the regard of the next theorem we observe that for a given function $\phi : [-1, 0] \to \boldsymbol{R}^{n \times n}$ of bounded variation and a matrix measure $\mu$, through integrations by parts the following relations hold for $k = 0, 1$:

$$\begin{aligned}\int_{-1}^{0} \exp(-\lambda r(\theta)) d(\mu \circ \phi_k)(\theta) \\ = P_k \mu(\Delta\phi) + \int_{-1}^{0} \lambda \exp(-\lambda r(\theta))(\mu \circ \phi_k)(\theta) dr(\theta),\end{aligned} \tag{33}$$

$$\begin{aligned}\int_{-1}^{0} \exp(\lambda \tau(\theta)) d(\mu \circ \phi_k)(\theta) \\ = Q_k \mu(\Delta\phi) - \int_{-1}^{0} \lambda \exp(\lambda \tau(\theta))(\mu \circ \phi_k)(\theta) d\tau(\theta),\end{aligned} \tag{34}$$

where $P_0 = -\exp(-\lambda r(-1))$, $P_1 = \exp(-\lambda r(0))$, $Q_0 = -\exp(\lambda \tau(-1))$, $Q_1 = \exp(\lambda \tau(0))$.

Analogously

$$\begin{aligned}\int_{-1}^{0} \exp(-\lambda r(\theta)) d(\mu \circ (-\phi_k))(\theta) \\ = P_k \mu(-\Delta\phi) + \int_{-1}^{0} \lambda \exp(-\lambda r(\theta))(\mu \circ (-\phi_k))(\theta) dr(\theta),\end{aligned} \tag{35}$$

and

$$\int_{-1}^{0} \exp(\lambda\tau(\theta))d(\mu\circ(-\phi_k))(\theta) \tag{36}$$

$$= Q_k\mu(-\Delta\phi) - \int_{-1}^{0} \lambda \exp(\lambda\tau(\theta))(\mu\circ(-\phi_k))(\theta)d\tau(\theta).$$

**Theorem 9.** a) *Let*

$$(\mu\circ v_1)(\theta) < 0 \quad \textit{and} \quad (\mu\circ\eta_1)(\theta) < 0, \qquad \textit{on } ]-1,0] \tag{37}$$

$$(\mu\circ v_0)(\theta) < 0 \quad \textit{and} \quad (\mu\circ\eta_0)(\theta) < 0, \qquad \textit{on } [-1,0[. \tag{38}$$

*If*

$$er(-1)\mu(\Delta v) + \int_{-1}^{0} (\mu\circ v_0)(\theta)dr(\theta) < -1, \tag{39}$$

*then* (2) *is oscillatory independently of the advances* $\tau(\theta)$.

b) *Let*

$$(\mu\circ(-v_1))(\theta) < 0 \quad \textit{and} \quad (\mu\circ(-\eta_1))(\theta) < 0, \qquad \textit{on } ]-1,0] \tag{40}$$

$$(\mu\circ(-v_0))(\theta) < 0 \quad \textit{and} \quad (\mu\circ(-\eta_0))(\theta) < 0, \qquad \textit{on } [-1,0[. \tag{41}$$

*If*

$$e\tau(-1)\mu(-\Delta\eta) + \int_{-1}^{0} (\mu\circ(-\eta_0))(\theta)d\tau(\theta) < -1, \tag{42}$$

*then* (2) *is oscillatory independently of the delays* $r(\theta)$.

*Proof.* i) Conditions (37) and (38) imply

$$\mu(\Delta v) = \mu(v_1(0)) = \mu(v_0(-1)) < 0,$$
$$\mu(\Delta\eta) = \mu(\eta_1(0)) = \mu(\eta_0(-1)) < 0,$$

and by (16), (8) is satisfied for $\lambda = 0$. In the same way, as by (40) and (41)

$$\mu(-\Delta v) = (\mu\circ(-v_1))(0) = (\mu\circ(-v_0))(-1) < 0$$
$$\mu(-\Delta\eta) = \mu\circ(-\eta_1)(0) = \mu\circ(-\eta_0)(-1) < 0,$$

through (21), one concludes that (9) is verified for $\lambda = 0$.

ii) Letting $\lambda > 0$ from (19), (33) and (34) one has

$$\mu(M(\lambda)) \le \exp(-\lambda r(0))\mu(\Delta v) + \int_{-1}^{0} \lambda \exp(-\lambda r(\theta))(\mu \circ v_1)(\theta) dr(\theta) \tag{43}$$
$$+ \exp(\lambda \tau(-1))\mu(\Delta \eta) + \int_{-1}^{0} \lambda \exp(\lambda \tau(\theta))(\mu \circ \eta_0)(\theta) d\tau(\theta).$$

Then by (37) and (38), we conclude that $\mu(M(\lambda)) < 0 < \lambda$.

In a similar way, when $\lambda < 0$ from the inequality (20) we obtain through (33) and (34),

$$\mu(M(\lambda)) \le \exp(-\lambda r(-1))\mu(\Delta v) - \int_{-1}^{0} \lambda \exp(-\lambda r(\theta))(\mu \circ v_0)(\theta) dr(\theta) \tag{44}$$
$$+ \exp(\lambda \tau(0))\mu(\Delta \eta) - \int_{-1}^{0} \lambda \exp(\lambda \tau(\theta))(\mu \circ \eta_1)(\theta) d\tau(\theta).$$

Therefore

$$\mu(M(\lambda)) \le \exp(-\lambda r(-1))\mu(\Delta v) - \lambda \int_{-1}^{0} \exp(-\lambda r(\theta))(\mu \circ v_0)(\theta) dr(\theta),$$

and since $\exp(-\lambda r(\theta)) \ge 1$ and $\exp(-\lambda r(-1)) \ge -e\lambda r(-1)$ we have then by (39)

$$\mu(M(\lambda)) \le -\lambda\left(er(-1)\mu(\Delta v) + \int_{-1}^{0} (\mu \circ v_0)(\theta) dr(\theta)\right) < \lambda. \tag{45}$$

Thus (8) holds.

iii) For $\lambda < 0$ one has by (24), (35) and (36)

$$\mu(-M(\lambda)) \le \exp(-\lambda r(-1))\mu(-\Delta v) \tag{46}$$
$$- \int_{-1}^{0} \lambda \exp(-\lambda r(\theta))(\mu \circ (-v_0))(\theta) dr(\theta)$$
$$+ \exp(\lambda \tau(0))\mu(-\Delta \eta) - \int_{-1}^{0} \lambda \exp(\lambda \tau(\theta))(\mu \circ (-\eta_1))(\theta) d\tau(\theta).$$

So, by (40) and (41) it is $\mu(-M(\lambda)) \le 0 < -\lambda$ for every $\lambda < 0$.

For $\lambda > 0$, one obtains analogously by (25), (35) and (36)

$$\mu(-M(\lambda)) \le \exp(-\lambda r(0))\mu(-\Delta v) \tag{47}$$
$$+ \int_{-1}^{0} \lambda \exp(-\lambda r(\theta))(\mu \circ (-v_1))(\theta) dr(\theta)$$
$$+ \exp(\lambda \tau(-1))\mu(-\Delta \eta)$$
$$+ \int_{-1}^{0} \lambda \exp(\lambda \tau(\theta))(\mu \circ (-\eta_0))(\theta) d\tau(\theta).$$

Therefore, since $\exp(\lambda\tau(\theta)) \geq 1$ and $\exp(\lambda\tau(-1)) \geq e\lambda\tau(-1)$ one has for every $\lambda > 0$

$$\mu(-M(\lambda)) \leq \lambda\left(e\tau(-1)\mu(-\Delta\eta) + \int_{-1}^{0}(\mu \circ (-\eta_0))(\theta)d\tau(\theta)\right).$$

Then by (42) $\mu(-M(\lambda)) < -\lambda$ and (9) is verified. ∎

In the regard of (1), one obtains the following corollary.

**Corollary 10.** a) *If with* $r_1 < \cdots < r_p$,

$$\mu\left(\sum_{i=1}^{k} A_i\right) < 0 \ \textit{and} \ \mu\left(\sum_{j=1}^{\ell} B_j\right) < 0, \quad k = 1, \ldots, p, \ \ell = 1, \ldots, q, \tag{48}$$

$$\mu\left(\sum_{i=k}^{p} A_i\right) < 0 \ \textit{and} \ \mu\left(\sum_{j=\ell}^{q} B_j\right) < 0, \quad k = 1, \ldots, p, \ \ell = 1, \ldots, q, \tag{49}$$

*and*

$$er_1\mu\left(\sum_{i=1}^{p} A_i\right) + \sum_{k=1}^{p-1}\mu\left(\sum_{i=k+1}^{p} A_i\right)(r_{k+1} - r_k) < -1, \tag{50}$$

*then* (4) *is oscillatory independently of the advances* $\tau_j \in \boldsymbol{R}^+$ *such that* $\tau_1 < \cdots < \tau_q$.

b) *With* $\tau_1 < \cdots < \tau_q$, *if*

$$\mu\left(-\sum_{i=1}^{k} A_i\right) < 0 \ \textit{and} \ \mu\left(-\sum_{j=1}^{\ell} B_j\right) < 0, \quad k = 1, \ldots, p, \ \ell = 1, \ldots, q, \tag{51}$$

$$\mu\left(-\sum_{i=k}^{p} A_i\right) < 0 \ \textit{and} \ \mu\left(-\sum_{j=\ell}^{q} B_j\right) < 0, \quad k = 1, \ldots, p, \ \ell = 1, \ldots, q, \tag{52}$$

*and*

$$e\tau_1\mu\left(\sum_{j=1}^{q} B_j\right) + \sum_{k=1}^{q-1}\mu\left(-\sum_{j=k+1}^{q} B_j\right)(\tau_{k+1} - \tau_k) < -1, \tag{53}$$

*then* (4) *is oscillatory independently of the delays* $r_j \in \boldsymbol{R}^+$ *such that* $r_1 < \cdots < r_p$.

*Proof.* a) Notice that,

$$(\mu \circ \nu_1)(\theta) = \mu\left(\sum_{i=1}^{p} H(\theta - \theta_i)A_i\right), \qquad (\mu \circ \nu_0)(\theta) = \mu\left(\sum_{i=1}^{p}(1 - H(\theta - \theta_i))A_i\right),$$

$$(\mu \circ \eta_1)(\theta) = \mu\left(\sum_{j=1}^{q} H(\theta - \xi_j)B_j\right), \qquad (\mu \circ \eta_0)(\theta) = \mu\left(\sum_{j=1}^{q}(1 - H(\theta - \xi_j))B_j\right),$$

and so (37) and (38) are fulfilled if (48) and (49) hold. On the other hand as

$$\int_{-1}^{0} (\mu \circ v_0)(\theta) dr(\theta) = \sum_{k=2}^{p} \mu\left(\sum_{i=k}^{p} A_i\right)(r_k - r_{k-1}),$$

(50) implies (39).

b) Now one has

$$(\mu \circ (-v_1))(\theta) = \mu\left(\sum_{i=1}^{p} H(\theta - \theta_i)(-A_i)\right),$$

$$(\mu \circ (-v_0))(\theta) = \mu\left(\sum_{i=1}^{p} (1 - H(\theta - \theta_i))(-A_i)\right),$$

$$(\mu \circ (-\eta_1))(\theta) = \mu\left(\sum_{j=1}^{q} H(\theta - \xi_j)(-B_j)\right),$$

$$(\mu \circ (-v_0))(\theta) = \mu\left(\sum_{j=1}^{q} (1 - H(\theta - \xi_j))(-B_j)\right).$$

Therefore (51) and (52) imply (40) and (41) and since

$$\int_{-1}^{0} (\mu \circ (-\eta_0))(\theta) d\tau(\theta) = \sum_{k=2}^{q} \mu\left(-\sum_{i=k}^{q} B_j\right)(\tau_k - \tau_{k-1}),$$

(53) implies (42). ■

The following example illustrates this corollary by using the matrix measure $\mu_\infty$ of a matrix $C = [c_{jk}]$ given by

$$\mu_\infty(C) = \max_{1 \leqslant j \leqslant n} \left\{ c_{jj} + \sum_{k \neq j} |c_{jk}| \right\}.$$

*Example* 11. Considering the system (1) with $r_1 = 1/10$, $r_2 = 1/2$ and

$$A_1 = \begin{bmatrix} -2 & 1 \\ -1 & -8 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -3 & -2 \\ 1 & -5 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -5 & -1 \\ -3 & -6 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -7 & -4 \\ 3 & -5 \end{bmatrix},$$

Using $\mu_\infty$, we have

$$\mu_\infty(A_1) = \max\{-1, -7\} = -1,$$

$$\mu_\infty(A_2) = \max\{-1, -4\} = -1,$$

$$\mu_\infty(A_1 + A_2) = \max\{-4, -13\} = -4,$$

$$\mu_\infty(B_1) = \max\{-4, -3\} = -3,$$

$$\mu_\infty(B_1 + B_2) = \max\{-7, -11\} = -7,$$

$$\mu_\infty(B_2) = \max\{-3, -2\} = -2.$$

Since

$$er_1\mu_\infty\left(\sum_{i=1}^{2} A_i\right) + \mu_\infty(A_1)(r_2 - r_1) = -\frac{2e}{5} - \frac{2}{5} < -1,$$

by the Corollary 10 a) the correspondent (1) is oscillatory independently of the advances $0 < \tau_1 < \tau_2$.

## 4. Positive delays and advances

If the delay and advance functions, $r(\theta)$ and $\tau(\theta)$, are both differentiable and positive one can still obtain the following theorem.

**Theorem 12.** a) *Let* (37) *and* (38) *hold. If*

$$1 + \ln(r(-1)|\mu(\Delta v)|) + r(-1)e^{-1}\int_{-1}^{0} (\mu \circ v_0)(\theta) d \ln r(\theta) > 0 \tag{54}$$

*then* (2) *is oscillatory independently of the advances* $\tau(\theta)$.

b) *Let* (40) *and* (41) *hold. If*

$$1 + \ln(\tau(-1)|\mu(-\Delta\eta)|) + \tau(-1)e^{-1}\int_{-1}^{0} (\mu \circ (-\eta_0))(\theta) d \ln \tau(\theta) > 0, \tag{55}$$

*then* (2) *is oscillatory independently of the delays* $r(\theta)$.

*Proof.* a) One can follow i) and ii) of the proof of the Theorem 9. Now the inequality (45) can be written as

$$\mu(M(\lambda)) \le \exp(-\lambda r(-1))\mu(\Delta v) - \int_{-1}^{0} \lambda r(\theta) \exp(-\lambda r(\theta))(\mu \circ v_0)(\theta) d \ln r(\theta),$$

and since $u\exp(-u) \le e^{-1}$, one obtains

$$\mu(M(\lambda)) \le \exp(-\lambda r(-1))\mu(\Delta v) - e^{-1}\int_{-1}^{0}(\mu\circ v_0)(\theta)d\ln r(\theta).$$

As the function

$$f(\lambda) = \exp(-\lambda r(-1))\mu(\Delta v) - \lambda$$

attains its maximum at

$$\lambda_0 = \frac{\ln(r(-1)|\mu(\Delta v)|)}{r(-1)}$$

and

$$f(\lambda_0) = -\frac{1+\ln(r(-1)|\mu(\Delta v)|)}{r(-1)},$$

(54) implies that

$$\mu(M(\lambda)) < \lambda$$

for every $\lambda < 0$.

b) Analogously, following i) and iii) of the proof of the Theorem 9, from the inequality (47) one obtains for $\lambda > 0$,

$$\begin{aligned}\mu(-M(\lambda)) &\le \exp(\lambda\tau(-1))\mu(-\Delta\eta) + \int_{-1}^{0}\lambda\tau(\theta)\exp(\lambda\tau(\theta))(\mu\circ(-\eta_0))(\theta)d\ln\tau(\theta)\\ &\le \exp(\lambda\tau(-1))\mu(-\Delta\eta) - e^{-1}\int_{-1}^{0}(\mu\circ(-\eta_0))(\theta)d\ln\tau(\theta).\end{aligned}$$

Taking the function

$$g(\lambda) = \exp(\lambda\tau(-1))\mu(-\Delta\eta) + \lambda,$$

its maximum is obtained at

$$\lambda_0 = -\frac{\ln(\tau(-1)|\mu(-\Delta\eta)|)}{\tau(-1)}$$

and

$$g(\lambda_0) = -\frac{1+\ln(\tau(-1)|\mu(-\Delta\eta)|)}{\tau(-1)}.$$

As (55) is equivalent to

$$g(\lambda_0) < e^{-1} \int_{-1}^{0} (\mu \circ (-\eta_0))(\theta) d \ln \tau(\theta),$$

one conclude then that $\mu(-M(\lambda)) < -\lambda$, for every real $\lambda > 0$. ∎

As before, the following corollary can be stated.

**Corollary 13.** a) *If with* $r_1 < \cdots < r_p$, (48) *and* (49) *hold and*

$$1 + \ln\left( r_1 \mu\left( \sum_{i=1}^{p} A_i \right)\right) + r_1 e^{-1} \sum_{k=1}^{p-1} \mu\left( \sum_{i=k+1}^{p} A_i \right) \ln \frac{r_{k+1}}{r_k} > 0 \tag{56}$$

*then* (1) *is oscillatory independently of the advances* $\tau_j \in \boldsymbol{R}^+$ *such that* $\tau_1 < \cdots < \tau_q$.

b) *With* $\tau_1 < \cdots < \tau_q$, *if* (51) *and* (52) *hold and*

$$1 + \ln\left( \tau_1 \mu\left( \sum_{j=1}^{q} B_j \right)\right) + \tau_1 e^{-1} \sum_{k=1}^{q-1} \mu\left( - \sum_{j=k+1}^{q} B_j \right) \ln \frac{\tau_{k+1}}{\tau_k} > 0, \tag{57}$$

*then* (1) *is oscillatory independently of the delays* $r_j \in \boldsymbol{R}^+$ *such that* $r_1 < \cdots < r_p$.

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