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Nonoscillations in retarded systems

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Abstract

This note is concerned with the existence of nonoscillatory solutions of a linear retarded system. Several criteria for nonoscillations are obtained, some of them regarding specific classes of continuous and differentiable delay functions.

Keywords: Difference retarded functional system; Delay difference systems; Nonoscillatory

1. Introduction

This work regards the existence of nonoscillations in the difference retarded functional system

$$x(t)+\int_{-1}^{0} d\big[\nu(\theta)\big]x\big(t-r(\theta)\big)=0, \tag{1}$$

where $x(t)\in\mathbb{R}^n$, $r(\theta)$ is a real continuous and positive function on $[-1,0]$, and $\nu(\theta)$ is a real n-by-n matrix valued function of bounded variation on $[-1,0]$.

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It will be also considered the relevant class of delay difference systems

$$x(t)+\sum_{j=1}^{p}A_jx(t-r_j)=0, \tag{2}$$

where the A_j are n-by-n real matrices and the r_j are positive real numbers such that $r_1<\cdots<r_p$. As is well known, these systems can be obtained, from (1), under the assumption that $\nu(\theta)$ is a step function with a number p of jump points. Denoting by H the Heaviside function, $\nu(\theta)$ can be given, for example, by

$$\nu(\theta)=\sum_{j=1}^{p}H(\theta-\theta_j)A_j, \tag{3}$$

for $-1<\theta_1<\cdots<\theta_p\leqslant 0$, where the delays, r_j, are obtained through any function, $r(\theta)$, continuous and positive on $[-1,0]$, which satisfy $r(\theta_j)=r_j$, for $j=1,\dots,p$.

Considering the value $\|r\|=\max\{r(\theta)\colon -1\leqslant\theta\leqslant 0\}$, a continuous function $x:[-\|r\|,+\infty[\to R$, is said a solution of (1) if satisfies this equation for every $t\geqslant 0$. A solution of (1), $x(t)=[x_1(t),\dots,x_n(t)]^T$, is called *oscillatory* if every component, $x_j(t)$, $j=1,\dots,n$, has arbitrary large zeros. Whenever all solutions of (1) are oscillatory, we will say that (1) is an *oscillatory* system. Otherwise, (1) is said *nonoscillatory*.

We will say that a function $\phi:[-1,0]\to\mathbb{R}$ is nondecreasing (nonincreasing) on $J\subset[-1,0]$, if for every $\theta_1,\theta_2\in J$ such that $\theta_1<\theta_2$, one has $\phi(\theta_1)\leqslant\phi(\theta_2)$ (respectively $\phi(\theta_2)\leqslant\phi(\theta_1)$); if ϕ is nondecreasing (nonincreasing) and nonconstant on $J\subset[-1,0]$, it will be called increasing (respectively decreasing) on J. If for every $\varepsilon>0$, sufficiently small, ϕ is increasing (decreasing) in $[\theta-\varepsilon,\theta+\varepsilon]$ ($[-\varepsilon,0]$ if $\theta=0$, $[-1,-1+\varepsilon]$ if $\theta=-1$) we will say that θ is a point of increase (respectively a point of decrease) of ϕ.

Letting $\nu(\theta)=[\nu_{jk}(\theta)]$ $(j,k=1,\dots,n)$ and $\theta_0\in[-1,0]$ such that $r(\theta_0)=\|r\|$, assuming that $r(\theta)<\|r\|$ for every $\theta\neq\theta_0$, in [1] is proven that (1) becomes nonoscillatory when θ_0 is a point of decrease of all the functions $\nu_{jk}(\theta)$. Moreover, it is shown that system (2) is nonoscillatory whenever all the entries of the matrix A_k are negative, where the index k is determined through the relation $r_k=\max\{r_j\colon j=1,\dots,p\}$.

Here we will show that (1) and (2) can be nonoscillatory in a different framework. On this purpose, matrix measures, already considered by several authors on the oscillation theory of delay systems, will play an useful role. For a matter of completeness we will report briefly, in the following, its definition and main properties.

For an induced norm, $\|\cdot\|$, in $\mathbb{M}_n(\mathbb{R})$, we associate a matrix measure $\mu:\mathbb{M}_n(\mathbb{R})\to\mathbb{R}$, which is defined for any $C\in\mathbb{M}_n(\mathbb{R})$ as

$$\mu(C)=\lim_{\gamma\to 0^+}\frac{\|I+\gamma C\|-1}{\gamma},$$

where by I we mean the identity matrix. Notice that, for every matrix measure, one has $\mu(0)=0$, $\mu(\pm I)=\pm 1$ and for the case $n=1$, $\mu(c)=c$ for every real number c.

Examples of matrix measures of a matrix $C=[c_{jk}]\in\mathbb{M}_n(\mathbb{R})$, are given by

$$\mu_1(C)=\max_{1\leqslant k\leqslant n}\left\{c_{kk}+\sum_{j\neq k}|c_{jk}|\right\},\qquad \mu_\infty(C)=\max_{1\leqslant j\leqslant n}\left\{c_{jj}+\sum_{k\neq j}|c_{jk}|\right\},$$

which correspond, respectively, to the induced norms in $\mathbb{M}_n(\mathbb{R})$,

$$\|C\|_1 = \max_{1\leqslant k\leqslant n}\left\{\sum_{j=1}^{n}|c_{jk}|\right\}, \qquad \|C\|_\infty = \max_{1\leqslant j\leqslant n}\left\{\sum_{k=1}^{n}|c_{jk}|\right\}.$$

Independently of the considered induced norm in $\mathbb{M}_n(\mathbb{R})$, a matrix measure, μ, has always the following properties (see [2]) for any $C \in \mathbb{M}_n(\mathbb{R})$:

(i) $\mu(\gamma C) = \gamma\mu(C)$, for every $\gamma \geqslant 0$;
(ii) $\mu(C_1) - \mu(-C_2) \leqslant \mu(C_1 + C_2) \leqslant \mu(C_1) + \mu(C_2)$ $(C_1, C_2 \in \mathbb{M}_n(\mathbb{R}))$;
(iii) $-\mu(-C) \leqslant \operatorname{Re} z \leqslant \mu(C)$, for every $z \in \sigma(C)$,

where by $\sigma(C)$ we denote the spectrum of the matrix C.

As an immediate consequence of the property (iii) above, through an argument used in [3] and [4], one has the following relationship between a matrix measure of a matrix and its determinant:

(iv) $\mu(C) \leqslant 0 \Rightarrow \det(C) \leqslant 0$, if n is odd;
(v) $\mu(C) \leqslant 0 \Rightarrow \det(C) \geqslant 0$, if n is even.

Another important property of any matrix measure regards the fact (see [5]) that $\mu \circ \eta$ is a real function of bounded variation on $[a, b]$, if η is a real n-by-n matrix valued function of bounded variation on an interval $[a, b]$. Moreover, the following inequalities hold:

(vi) If $\phi \in C([a, b]; \mathbb{R})$ is nonincreasing and positive, then

$$\mu\left(\int_a^b \phi(\theta)\, d\big[\eta(\theta)\big]\right) \leqslant \int_a^b \phi(\theta)\, d\big(\mu\big(\eta(\theta) - \eta(a)\big)\big).$$

(vii) If $\phi \in C([a, b]; \mathbb{R})$ is nondecreasing and positive, then

$$\mu\left(\int_a^b \phi(\theta)\, d\big[\eta(\theta)\big]\right) \leqslant -\int_a^b \phi(\theta)\, d\big(\mu\big(\eta(b) - \eta(\theta)\big)\big).$$

For a given real n-by-n matrix valued function, η, of bounded variation on the interval $[-1, 0]$, these properties give relevance to that the following functions η_0 and η_1 be considered:

$$\eta_0(\theta) = \eta(0) - \eta(\theta), \qquad \eta_1(\theta) = \eta(\theta) - \eta(-1) \quad \big(\theta \in [-1, 0]\big).$$

By $\Delta\eta$ we will denote the difference $\eta(0) - \eta(-1) = \eta_0(-1) = \eta_1(0)$.

Letting

$$A(\lambda) = \int_{-1}^{0} \exp\big(-\lambda r(\theta)\big)\, d\big[\nu(\theta)\big],$$

according to [6], we recall that denoting by I the n-by-n identity matrix, the system (1) is nonoscillatory if and only if there exists a real λ such that

$$\det\big(I + A(\lambda)\big) = 0,$$

that is if and only if

$$\exists\lambda \in \mathbb{R}: \quad 1 \in \sigma\big(-A(\lambda)\big). \tag{4}$$

Taking the real function

$$s(\lambda) = \max\big\{\operatorname{Re} z\colon\ z \in \sigma\big(-A(\lambda)\big)\big\}, \tag{5}$$

and assuming that

$$s(\lambda) \in \sigma\big(-A(\lambda)\big), \quad \forall\lambda \in \mathbb{R}, \tag{6}$$

one has by [1, Theorem 1] that (1) is nonoscillatory if and only if there exists a real λ_0 such that $s(\lambda_0) \geqslant 1$.

We recall that (6) is satisfied when, for each real λ, the matrix $-A(\lambda)$ is essentially nonnegative—that is, when at least its off-diagonal entries are nonnegative. This occurs when, at least, the off-diagonal functions $\nu_{jk}(\theta)$ $(j \neq k)$, of $\nu(\theta) = [\nu_{jk}(\theta)]$, are nonincreasing functions on $[-1, 0]$. Assumption (6) also holds when for every $\theta \in [-1, 0]$, $\nu(\theta)$ are symmetric or triangular matrices.

Thus assuming (6), by property (iii) of the matrix measures, we have that if

$$\exists\lambda_0 \in \mathbb{R}: \quad \mu\big(A(\lambda_0)\big) \leqslant -1, \tag{7}$$

then (1) is nonoscillatory.

We make notice that (7) is also a sufficient condition for nonoscillations, when the order, n, of (1) is an odd integer. In fact, (7) implies that

$$\mu\big(I + A(\lambda_0)\big) \leqslant 1 + \mu\big(A(\lambda_0)\big) \leqslant 0,$$

and if n is an odd integer, by property (iv) of the matrix measures one has necessarily $\det(I + A(\lambda_0)) \leqslant 0$. Then since $\det(I + A(\lambda)) \to 1$, as $\lambda \to +\infty$, we conclude that $\det(I + A(\lambda)) = 0$ for some real λ and consequently that (1) is nonoscillatory.

In the following sections we will implicitly assume that either hypothesis (6) holds or n is an odd integer.

2. Nonoscillations for classes of continuous delays

Denote by C^+ the set of all real continuous and positive functions on $[-1, 0]$. In this section we start by obtaining several criteria of nonoscillations regarding some families of delays in C^+.

Theorem 1. *If*

$$\mu(\Delta\nu) \leqslant -1, \tag{8}$$

then (1) *is nonoscillatory for all delay functions in* C^+.

Proof. As $A(0) = \Delta\nu$, under (8) one has (7) satisfied and the theorem follows. $\square$

Therefore for system (2) we obtain the following corollary.

Corollary 2. *If*

$$\sum_{j=1}^{p} \mu(A_j) \leqslant -1,$$

then (2) *is nonoscillatory for every* $(r_1, \dots, r_p)$ *in* $\mathbb{R}_+^p$ *such that* $r_1 < \dots < r_p$.

Assuming that $-1 \leqslant \alpha \leqslant \beta \leqslant 0$, let $C^+(\alpha, \beta)$ be the family of all functions in C^+, which are increasing on $[-1, \alpha]$, constant on $[\alpha, \beta]$ and decreasing on $[\beta, 0]$.

Theorem 3. *If*

$$(\mu \circ \nu_1) \quad \text{is nondecreasing on} \quad [-1, \alpha], \tag{9}$$

$$(\mu \circ \nu_0) \quad \text{is nonincreasing on} \quad [\beta, 0], \tag{10}$$

and

$$(\mu \circ \nu_0)(\beta) + \mu\big(\nu(\beta) - \nu(\alpha)\big) + (\mu \circ \nu_1)(\alpha) \leqslant -1, \tag{11}$$

then (1) *is nonoscillatory for every delay function in* $C^+(\alpha, \beta)$.

Proof. By properties (i) and (ii) of the matrix measures, we have

$$\mu\big(A(\lambda)\big) \leqslant \mu\left(\int_{-1}^{\alpha} \exp\big(-\lambda r(\theta)\big)\, d\big[\nu(\theta)\big]\right) + \exp\big(-\lambda r(\alpha)\big)\mu\big(\nu(\beta) - \nu(\alpha)\big)$$
$$+ \mu\left(\int_{\beta}^{0} \exp\big(-\lambda r(\theta)\big)\, d\big[\nu(\theta)\big]\right). \tag{12}$$

For $\lambda > 0$, properties (vi) and (vii) imply that

$$\mu\big(A(\lambda)\big) \leqslant \int_{-1}^{\alpha} \exp\big(-\lambda r(\theta)\big)\, d(\mu \circ \nu_1)(\theta) + \exp\big(-\lambda r(\alpha)\big)\mu\big(\nu(\beta) - \nu(\alpha)\big)$$
$$- \int_{\beta}^{0} \exp\big(-\lambda r(\theta)\big)\, d(\mu \circ \nu_0)(\theta). \tag{13}$$

Then by assumptions (9) and (10) we have

$$\mu\big(A(\lambda)\big) \leqslant (\mu \circ \nu_1)(\alpha) + \mu\big(\nu(\beta) - \nu(\alpha)\big) + (\mu \circ \nu_0)(\beta),$$

and by (11) we conclude that, for every $\lambda > 0$, $\mu(A(\lambda)) \leqslant -1$. Hence by (7) one has (1) nonoscillatory for every delay function in $C^+(\alpha, \beta)$. $\square$

Theorem 4. *If*

$$\mu\big(\nu(\alpha)-\nu(\theta)\big) \quad \textit{is nonincreasing} \quad \textit{on } [-1,\alpha], \tag{14}$$

$$\mu\big(\nu(\theta)-\nu(\beta)\big) \quad \textit{is nondecreasing} \quad \textit{on } [\beta,0], \tag{15}$$

and (11) *is satisfied, then* (1) *is nonoscillatory for every delay function in* $C^+(\alpha,\beta)$.

Proof. From (12), properties (vi) and (vii) imply that, for $\lambda<0$,

$$\mu\big(A(\lambda)\big) \leqslant -\int_{-1}^{\alpha} \exp\big(-\lambda r(\theta)\big)\, d\mu\big(\nu(\alpha)-\nu(\theta)\big) + \exp\big(-\lambda r(\alpha)\big)\mu\big(\nu(\beta)-\nu(\alpha)\big) + \int_{\beta}^{0} \exp\big(-\lambda r(\theta)\big)\, d\mu\big(\nu(\theta)-\nu(\beta)\big). \tag{16}$$

Since $\exp(-\lambda r(\theta)) \leqslant \exp(-\lambda r(\alpha))$ for every $\theta \in [-1,\alpha]\cup[\beta,0]$, by (14) and (15) we obtain

$$\mu\big(A(\lambda)\big) \leqslant \exp\big(-\lambda r(\alpha)\big)(\mu\circ\nu_1)(\alpha) + \exp\big(-\lambda r(\alpha)\big)\mu\big(\nu(\beta)-\nu(\alpha)\big) + \exp\big(-\lambda r(\alpha)\big)(\mu\circ\nu_0)(\beta).$$

Then by (11) we have, for every $\lambda<0$, $\mu(A(\lambda)) \leqslant -1$. Hence by (7) one has (1) nonoscillatory for every delay function in $C^+(\alpha,\beta)$. $\square$

Remark 5. Notice that each one of the conditions (9) and (14), implies $(\mu\circ\nu_1)(\alpha) \geqslant 0$ and each one of the assumptions (10) or (15) implies $(\mu\circ\nu_0)(\beta) \geqslant 0$. Therefore, in both theorems we have to exclude the possibility of having $\alpha=\beta$, since in order to have the inequality (11) satisfied, the term $\mu(\nu(\beta)-\nu(\alpha))$ must have a large preponderance. This fact introduces some difficulty in the application of the Theorems 3 and 4 as it can be observed through the following example.

Example 6. Consider (1) with

$$\nu(\theta) = \begin{bmatrix} \frac{100}{3}\theta^3 + 50\theta^2 + 9\theta & -\theta+2 \\ -\theta+1 & -8\theta-1 \end{bmatrix}.$$

For the matrix measure μ_∞ we have

$$\begin{aligned}
&(\mu_\infty\circ\nu_0)\left(-\frac{1}{10}\right) + \mu_\infty\left(\nu\left(-\frac{1}{10}\right) - \nu\left(-\frac{9}{10}\right)\right) + (\mu_\infty\circ\nu_1)\left(-\frac{9}{10}\right) \\
&\quad = \mu_\infty\left(\begin{bmatrix} \frac{13}{30} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{4}{5} \end{bmatrix}\right) + \mu_\infty\left(\begin{bmatrix} -\frac{128}{15} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{32}{5} \end{bmatrix}\right) + \mu_\infty\left(\begin{bmatrix} \frac{13}{30} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{4}{5} \end{bmatrix}\right) \\
&\quad = \frac{8}{15} - \frac{28}{5} + \frac{8}{15} = -\frac{68}{15}.
\end{aligned}$$

For $\theta \in \left[-1, -\frac{9}{10}\right]$,

$$
\begin{aligned}
(\mu_\infty \circ \nu_1)(\theta) &= \mu_\infty\left(\begin{bmatrix} \frac{100}{3}\theta^3 + 50\theta^2 + 9\theta - \frac{23}{3} & -\theta - 1 \\ -\theta - 1 & -8\theta - 8 \end{bmatrix}\right) \\
&= \max\left\{\frac{100}{3}\theta^3 + 50\theta^2 + 10\theta - \frac{20}{3}, -7\theta - 7\right\} \\
&= \frac{100}{3}\theta^3 + 50\theta^2 + 10\theta - \frac{20}{3}
\end{aligned}
$$

is increasing, and for $\theta \in \left[-\frac{1}{10}, 0\right]$,

$$
\begin{aligned}
(\mu_\infty \circ \nu_0)(\theta) &= \mu_\infty\left(\begin{bmatrix} -\frac{100}{3}\theta^3 - 50\theta^2 - 9\theta & \theta \\ \theta & 8\theta \end{bmatrix}\right) \\
&= \max\left\{-\frac{100}{3}\theta^3 - 50\theta^2 - 10\theta, 7\theta\right\} \\
&= -\frac{100}{3}\theta^3 - 50\theta^2 - 10\theta
\end{aligned}
$$

is decreasing. Hence by Theorem 3, the corresponding system (1) is nonoscillatory for every $r \in C^+\left(-\frac{9}{10}, -\frac{1}{10}\right)$.

Remark 7. Notice that for the scalar case of (1), that is for $n = 1$, the condition (11) in Theorems 3 and 4 gives the assumption (8) of Theorem 1. So in the scalar case only this theorem can be considered.

3. Nonoscillations for families of differentiable delays

Still assuming that $-1 \leqslant \alpha \leqslant \beta \leqslant 0$, let now $D^+(\alpha, \beta)$ be the family of all functions in $C^+(\alpha, \beta)$ which are differentiable on the interval $[-1, 0]$.

Theorem 8. *If*

$$\mu\big(\nu(\alpha) - \nu(\theta)\big) \leqslant 0, \quad \textit{for } \theta \in [-1, \alpha], \tag{17}$$

$$\mu\big(\nu(\theta) - \nu(\beta)\big) \leqslant 0, \quad \textit{for } \theta \in [\beta, 0], \tag{18}$$

$$\mu\big(\nu(\beta) - \nu(\alpha)\big) \leqslant 0, \tag{19}$$

and

$$\min\big\{(\mu \circ \nu_1)(\alpha), \mu\big(\nu(\beta) - \nu(\alpha)\big), (\mu \circ \nu_0)(\beta)\big\} < 0, \tag{20}$$

then (1) *is nonoscillatory for all delay functions in* $D^+(\alpha, \beta)$.

Proof. Let $r(\theta)$ be any delay function in $D^+(\alpha, \beta)$ and $\lambda < 0$. Integrating by parts each one of the integrals in the right-hand member of (16), we obtain

$$\mu\big(A(\lambda)\big) \leqslant \exp\big(-\lambda r(-1)\big)(\mu \circ \nu_1)(\alpha) + \exp\big(-\lambda r(\alpha)\big)\mu\big(\nu(\beta) - \nu(\alpha)\big)$$
$$+ \exp\big(-\lambda r(0)\big)(\mu \circ \nu_0)(\beta) - \int_{-1}^{\alpha} \lambda \exp\big(-\lambda r(\theta)\big)\mu\big(\nu(\alpha) - \nu(\theta)\big)\, dr(\theta)$$
$$+ \int_{\beta}^{0} \lambda \exp\big(-\lambda r(\theta)\big)\mu\big(\nu(\theta) - \nu(\beta)\big)\, dr(\theta).$$

Then the assumptions (17), (18) and $r \in D^+(\alpha, \beta)$ imply that

$$\mu\big(A(\lambda)\big) \leqslant \exp\big(-\lambda r(-1)\big)(\mu \circ \nu_1)(\alpha) + \exp\big(-\lambda r(\alpha)\big)\mu\big(\nu(\beta) - \nu(\alpha)\big)$$
$$+ \exp\big(-\lambda r(0)\big)(\mu \circ \nu_0)(\beta). \tag{21}$$

By (19) and (20), the right-hand member of (21) goes to $-\infty$, as $\lambda \to -\infty$. Then there exist $\lambda < 0$ such that $\mu(A(\lambda)) \leqslant -1$. Thus (1) is nonoscillatory for all delay functions in $D^+(\alpha, \beta)$. $\square$

The following example illustrates Theorem 8.

Example 9. Consider the system (1) with

$$\nu(\theta) = \begin{bmatrix} -\theta + \frac{3}{2} & -\theta - \frac{1}{4} & \frac{5}{4} \\ 2 & -4\theta - 3 & 1 \\ \frac{1}{2} & \theta + \frac{1}{4} & -\theta + \frac{1}{2} \end{bmatrix}.$$

Using the matrix measure μ_1, we have for $\theta \in \big(-1, -\frac{3}{4}\big]$,

$$\mu_1\left(\nu\left(-\frac{3}{4}\right) - \nu(\theta)\right) = \mu_1\left(\begin{bmatrix} \theta + \frac{3}{4} & \theta + \frac{3}{4} & 0 \\ 0 & 4\theta + 3 & 0 \\ 0 & -\theta - \frac{3}{4} & \theta + \frac{3}{4} \end{bmatrix}\right)$$
$$= \max\left\{\theta + \frac{3}{4}, 2\theta + \frac{3}{2}\right\}$$
$$= \theta + \frac{3}{4} \leqslant 0,$$

and for $\theta \in \big[-\frac{1}{4}, 0\big)$,

$$\mu_1\left(\nu(\theta) - \nu\left(-\frac{1}{4}\right)\right) = \mu_1\left(\begin{bmatrix} -\theta - \frac{1}{4} & -\theta - \frac{1}{4} & 0 \\ 0 & -4\theta - 1 & 0 \\ 0 & \theta + \frac{1}{4} & -\theta - \frac{1}{4} \end{bmatrix}\right)$$
$$= \max\left\{-\theta - \frac{1}{4}, -2\theta - \frac{1}{2}\right\}$$
$$= -\theta - \frac{1}{4} \leqslant 0.$$

On the other hand, as

$$\mu_1\left(\nu\left(-\frac{1}{4}\right)-\nu\left(-\frac{3}{4}\right)\right)=\mu_1\left(\begin{bmatrix}-\frac{1}{2} & -\frac{1}{2} & 0\\ 0 & -2 & 0\\ 0 & \frac{1}{2} & -\frac{1}{2}\end{bmatrix}\right)=-\frac{1}{2},$$

the corresponding system (1) is nonoscillatory for all delays in the class $D^+\left(-\frac{3}{4},-\frac{1}{4}\right)$.

In case of having $\alpha=\beta=\theta_0\in[-1,0]$, we obtain the family, $D^+(\theta_0)$, of all differentiable and positive functions which are increasing on $[-1,\theta_0]$ and decreasing on $[\theta_0,0]$. If $\theta_0=-1$, $D^+(-1)$ is the class of all positive, differentiable and decreasing functions on $[-1,0]$, which we will denote by D_d^+. For $\theta_0=0$, we obtain the family D_i^+ of all positive, differentiable and increasing functions on $[-1,0]$.

The following corollary holds.

Corollary 10. *If*

$$\mu\big(\nu(\theta_0)-\nu(\theta)\big)\leqslant 0,\quad \text{for } \theta\in[-1,\theta_0],$$
$$\mu\big(\nu(\theta)-\nu(\theta_0)\big)\leqslant 0,\quad \text{for } \theta\in[\theta_0,0],$$

and

$$\min\big\{(\mu\circ\nu_1)(\theta_0),(\mu\circ\nu_0)(\theta_0)\big\}<0,$$

then (1) *is nonoscillatory for all delay functions in* $D^+(\theta_0)$.

The cases $\theta_0=-1$ and $\theta_0=0$ give rise, respectively, to the two corollaries below.

Corollary 11. *If*

$$(\mu\circ\nu_1)(\theta)\leqslant 0,\quad \text{for } \theta\in[-1,0],$$

and

$$\mu(\Delta\nu)<0,$$

then (1) *is nonoscillatory for all delay functions in* D_d^+.

Corollary 12. *If*

$$(\mu\circ\nu_0)(\theta)\leqslant 0,\quad \text{for } \theta\in[-1,0],$$

and

$$\mu(\Delta\nu)<0,$$

then (1) *is nonoscillatory for all delay functions in* D_i^+.

In the following example we illustrate the Corollary 12.

Example 13. Let now

$$\nu(\theta)=\begin{bmatrix} -\frac{\theta}{2}-1 & 2 & 5 \\ \frac{\theta}{4} & -\frac{2\theta}{3}+9 & 0 \\ 2 & \frac{\theta}{3}-1 & -\frac{4\theta}{5}+6 \end{bmatrix}.$$

Using the matrix measure μ_∞, we have

$$(\mu_\infty\circ\nu_0)(\theta)=\mu_\infty\left(\begin{bmatrix} \frac{1}{2}\theta & 0 & 0 \\ -\frac{1}{4}\theta & \frac{2}{3}\theta & 0 \\ 0 & -\frac{1}{3}\theta & \frac{4}{5}\theta \end{bmatrix}\right)=\max\left\{\frac{\theta}{2},\frac{5}{12}\theta,\frac{7}{15}\theta\right\}\leqslant 0$$

for every $\theta\in[-1,0]$, and

$$\mu_\infty(\Delta\nu)=\mu_\infty\left(\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{4} & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{4}{5} \end{bmatrix}\right)=\max\left\{-\frac{1}{2},-\frac{5}{12},-\frac{7}{15}\right\}=-\frac{5}{12}.$$

Then by Corollary 12, the system is nonoscillatory for all delay functions in D_i^+.

The Corollary 12 applied to Eq. (2), enable us to obtain the following corollary.

Corollary 14. *Let $(r_1,r_2,\ldots,r_p)\in\mathbb{R}_+^p$ be such that $r_1<r_2<\cdots<r_p$. Then* (2) *is nonoscillatory if*

$$\mu\left(\sum_{k=j}^{p}A_k\right)\leqslant 0,\quad \textit{for every } 2\leqslant j\leqslant p,\quad \textit{and}\quad \mu\left(\sum_{k=1}^{p}A_k\right)<0.$$

Example 15. Consider the system

$$x(t)+A_1x(t-r_1)+A_2x(t-r_2)+A_3x(t-r_3)+A_4x(t-r_4)=0, \tag{22}$$

where

$$A_1=\begin{bmatrix} 4 & -2 & 0 \\ 0 & -3 & -1 \\ 2 & 0 & 2 \end{bmatrix},\qquad A_2=\begin{bmatrix} -2 & 0 & 1 \\ 0 & 4 & 2 \\ 2 & 1 & -2 \end{bmatrix},$$

$$A_3=\begin{bmatrix} -3 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & -1 & -2 \end{bmatrix}\quad\text{and}\quad A_4=\begin{bmatrix} -5 & 1 & 3 \\ 0 & -2 & -2 \\ -6 & -1 & -8 \end{bmatrix}.$$

We have

$$\begin{aligned}
\mu_\infty(A_4)&=\max\{-1,0,-1\}=0,\\
\mu_\infty(A_3+A_4)&=\max\{-3,-4,-2\}=-2,\\
\mu_\infty(A_2+A_3+A_4)&=\max\{-4,0,-7\}=0,\\
\mu_\infty(A_1+A_2+A_3+A_4)&=\max\{-2,-4,-7\}=-2.
\end{aligned}$$

Hence the system (22) is nonoscillatory for each family of delays $(r_1, r_2, r_3, r_4) \in \mathbb{R}_+^4$ such that $r_1 < r_2 < r_3 < r_4$.

Remark 16. Theorem 8 and all the corollaries obtained in this section can obviously be applied to the scalar case of (1) and (2), respectively. However in that case, the proof of Theorem 8 can be substantially simplified giving rise to [7, Theorem 1]. Moreover, the scalar case of (1) and (2) present specific characteristics as it can be seen in the first part of [7].

4. Mixed criteria for nonoscillations

In this section we will describe several results involving conditions on the delays, $r(\theta)$, and on the matrix function, $\nu(\theta)$, in order to have (1) nonoscillatory.

Theorem 17. *Let $r(\theta)$ be any delay function in $D^+(\alpha, \beta)$. If*

$$(\mu \circ \nu_1)(\theta) \geqslant 0, \quad \text{for every } \theta \in [-1, \alpha], \tag{23}$$

$$(\mu \circ \nu_0)(\theta) \geqslant 0, \quad \text{for every } \theta \in [\beta, 0], \tag{24}$$

and

$$\int_{-1}^{\alpha} (\mu \circ \nu_1)(\theta)\, d \ln r(\theta) - \int_{\beta}^{0} (\mu \circ \nu_0)(\theta)\, d \ln r(\theta) < -e\big[1 + (\mu \circ \nu_1)(\alpha) + \mu\big(\nu(\beta) - \nu(\alpha)\big) + (\mu \circ \nu_0)(\beta)\big], \tag{25}$$

then (1) *is nonoscillatory.*

Proof. Let $\lambda > 0$. Integrating by parts the integrals in (13), we obtain

$$\begin{aligned}\mu\big(A(\lambda)\big) \leqslant{}& \exp\big(-\lambda r(\alpha)\big)(\mu \circ \nu_1)(\alpha) + \int_{-1}^{\alpha} \lambda \exp\big(-\lambda r(\theta)\big)(\mu \circ \nu_1)(\theta)\, dr(\theta) \\ &+ \exp\big(-\lambda r(\alpha)\big)\mu\big(\nu(\beta) - \nu(\alpha)\big) + \exp\big(-\lambda r(\alpha)\big)(\mu \circ \nu_0)(\beta) \\ &- \int_{\beta}^{0} \lambda \exp\big(-\lambda r(\theta)\big)(\mu \circ \nu_0)(\theta)\, dr(\theta).\end{aligned} \tag{26}$$

Since $\lambda r(\theta) e^{-\lambda r(\theta)} \leqslant e^{-1}$ for every $\theta \in [-1, 0]$ and $\lambda \in \mathbb{R}$, by (23) and (24), we have

$$\begin{aligned}\mu\big(A(\lambda)\big) \leqslant{}& \exp\big(-\lambda r(\alpha)\big)\big[(\mu \circ \nu_1)(\alpha) + \mu\big(\nu(\beta) - \nu(\alpha)\big) + (\mu \circ \nu_0)(\beta)\big] \\ &+ e^{-1}\left[\int_{-1}^{\alpha} (\mu \circ \nu_1)(\theta)\, d \ln r(\theta) - \int_{\beta}^{0} (\mu \circ \nu_0)(\theta)\, d \ln r(\theta)\right].\end{aligned}$$

On the other hand, by (25), the function

$$f(\lambda) = -1 - \exp\bigl(-\lambda r(\alpha)\bigr)\bigl[(\mu \circ \nu_1)(\alpha) + \mu\bigl(\nu(\beta) - \nu(\alpha)\bigr) + (\mu \circ \nu_0)(\beta)\bigr]$$

is such that

$$ef(0) > \int_{-1}^{\alpha} (\mu \circ \nu_1)(\theta)\, d\ln r(\theta) - \int_{\beta}^{0} (\mu \circ \nu_0)(\theta)\, d\ln r(\theta) \geqslant 0.$$

Therefore $f(\lambda)$ decreases to -1, as $\lambda \to +\infty$, and consequently there exists a $\lambda_0 > 0$ such that

$$f(\lambda_0) = e^{-1}\left[\int_{-1}^{\alpha} (\mu \circ \nu_1)(\theta)\, d\ln r(\theta) - \int_{\beta}^{0} (\mu \circ \nu_0)(\theta)\, d\ln r(\theta)\right].$$

Hence $\mu(A(\lambda_0)) \leqslant -1$, which completes the proof. $\square$

In Theorem 17 it is not possible to have $\alpha = \beta$. In fact, in such circumstances, one easily sees that the assumption (25) is in contradiction with (23) and (24). This means that the theorem cannot be applied to (1), when $r(\theta)$ is in any class of functions of the type $D^+(\theta_0)$, even when $\theta_0 = -1$ or $\theta_0 = 0$.

However, with some changes in the proof of Theorem 17, for $(\mu \circ \nu_1)$ and $(\mu \circ \nu_0)$ both nonnegative, is possible to have a different situation.

Theorem 18. *Let $r(\theta)$ be in $D^+(\alpha, \beta)$. If*

$$(\mu \circ \nu_1)(\theta) \leqslant 0, \quad \text{for every } \theta \in [-1, \alpha], \tag{27}$$

$$(\mu \circ \nu_0)(\theta) \leqslant 0, \quad \text{for every } \theta \in [\beta, 0], \tag{28}$$

$$(\mu \circ \nu_1)(\alpha) + \mu\bigl(\nu(\beta) - \nu(\alpha)\bigr) + (\mu \circ \nu_0)(\beta) \leqslant 0 \tag{29}$$

and

$$\int_{-1}^{\alpha} (\mu \circ \nu_1)(\theta)\, dr(\theta) - \int_{\beta}^{0} (\mu \circ \nu_0)(\theta)\, dr(\theta) \leqslant -er(\alpha), \tag{30}$$

then (1) *is nonoscillatory.*

Proof. Let $\lambda > 0$. From (26) one has, by (27)–(29),

$$\begin{aligned}\mu\bigl(A(\lambda)\bigr) &\leqslant \exp\bigl(-\lambda r(\alpha)\bigr)\bigl[(\mu \circ \nu_1)(\alpha) + (\mu \circ \nu_0)(\beta) + \mu\bigl(\nu(\beta) - \nu(\alpha)\bigr)\bigr] \\ &\quad + \lambda \exp\bigl(-\lambda r(\alpha)\bigr)\left[\int_{-1}^{\alpha} (\mu \circ \nu_1)(\theta)\, dr(\theta) - \int_{\beta}^{0} (\mu \circ \nu_0)(\theta)\, dr(\theta)\right] \\ &\leqslant \lambda \exp\bigl(-\lambda r(\alpha)\bigr)\left[\int_{-1}^{\alpha} (\mu \circ \nu_1)(\theta)\, dr(\theta) - \int_{\beta}^{0} (\mu \circ \nu_0)(\theta)\, dr(\theta)\right].\end{aligned}$$

The function

$$g(\lambda) = \frac{\exp(\lambda r(\alpha))}{\lambda}$$

is such that $g(\lambda) \to +\infty$ as $\lambda \to +\infty$ or $\lambda \to 0^+$, and by (30),

$$\min\{g(\lambda):\ \lambda > 0\} = er(\alpha) \leqslant -\left[\int_{-1}^{\alpha} (\mu \circ \nu_1)(\theta)\, dr(\theta) - \int_{\beta}^{0} (\mu \circ \nu_0)(\theta)\, dr(\theta)\right].$$

Therefore, there exists a $\lambda_0 > 0$ in manner that

$$g(\lambda_0) = -\left[\int_{-1}^{\alpha} (\mu \circ \nu_1)(\theta)\, dr(\theta) - \int_{\beta}^{0} (\mu \circ \nu_0)(\theta)\, dr(\theta)\right],$$

that is such that

$$1 + \lambda_0 \exp\bigl(-\lambda_0 r(\alpha)\bigr)\left[\int_{-1}^{\alpha} (\mu \circ \nu_1)(\theta)\, dr(\theta) - \int_{\beta}^{0} (\mu \circ \nu_0)(\theta)\, dr(\theta)\right] = 0.$$

Thus $\mu(A(\lambda_0)) \leqslant -1$ and so (1) is a nonoscillatory system. $\square$

Considering the case $\alpha = \beta = \theta_0 \in [-1, 0]$ and, in particular, the cases $\theta_0 = 0$ and $\theta_0 = -1$, is possible to obtain, respectively, the corollaries described in the sequel.

Corollary 19. *Let $r(\theta)$ be in $D^+(\theta_0)$. If*

$$(\mu \circ \nu_1)(\theta) \leqslant 0, \quad \text{for every } \theta \in [-1, \theta_0], \tag{31}$$

$$(\mu \circ \nu_0)(\theta) \leqslant 0, \quad \text{for every } \theta \in [\theta_0, 0], \tag{32}$$

and

$$\int_{-1}^{\theta_0} (\mu \circ \nu_1)(\theta)\, dr(\theta) - \int_{\theta_0}^{0} (\mu \circ \nu_0)(\theta)\, dr(\theta) \leqslant -er(\theta_0), \tag{33}$$

then (1) *is nonoscillatory.*

Corollary 20. *If*

$$(\mu \circ \nu_1)(\theta) \leqslant 0, \quad \text{for every } \theta \in [-1, 0], \tag{34}$$

and $r(\theta)$ in D_i^+ is such that

$$\int_{-1}^{0} (\mu \circ \nu_1)(\theta)\, dr(\theta) \leqslant -er(0), \tag{35}$$

then (1) *is nonoscillatory.*

Corollary 21. *If*

$$(\mu \circ \nu_0)(\theta) \leqslant 0, \quad \textit{for every } \theta \in [-1, 0], \tag{36}$$

and $r(\theta)$ *in* D_d^+ *is such that*

$$\int_{-1}^{0} (\mu \circ \nu_0)(\theta)\, dr(\theta) \geqslant er(-1), \tag{37}$$

then (1) *is nonoscillatory.*

Example 22. Let (1) with $r(\theta) = -\theta + \varepsilon$ $(\varepsilon > 0)$ and

$$\nu(\theta) = \begin{bmatrix} -7\theta & \frac{\theta}{7} \\ \frac{\theta}{7} & -7\theta - 1 \end{bmatrix}.$$

As $\nu(\theta)$ is symmetric, using as matrix measure μ either μ_∞ or μ_1, we have

$$(\mu \circ \nu_0)(\theta) = \mu\left(\begin{bmatrix} 7\theta & -\frac{\theta}{7} \\ -\frac{\theta}{7} & 7\theta \end{bmatrix}\right) = 7\theta - \frac{\theta}{7} = \frac{48}{7}\theta \leqslant 0,$$

for every $\theta \in [-1, 0]$. Since

$$\int_{-1}^{0} (\mu \circ \nu_0)(\theta)\, dr(\theta) = -\int_{-1}^{0} \frac{48}{7}\theta\, d\theta = \frac{24}{7},$$

the corresponding system (1), by the Corollary 21, is nonoscillatory for every ε such that $0 < \varepsilon \leqslant \frac{24}{7e} - 1$.

For $\nu(\theta)$ given by (3) with $\theta_p = 0$, Corollary 20 enables us to conclude the following corollary, relative to the system (2).

Corollary 23. *If*

$$\mu\left(\sum_{k=1}^{j} A_k\right) \leqslant 0, \quad \textit{for } 1 \leqslant j \leqslant p, \quad \textit{and}$$

$$\sum_{j=1}^{p-1} \mu\left(\sum_{k=1}^{j} A_k\right)(r_{j+1} - r_j) \leqslant -er_p,$$

then (2) *is nonoscillatory.*

Example 24. Let the system

$$x(t) + A_1 x\left(t - \frac{1}{4}\right) + A_2 x\left(t - \frac{1}{2}\right) + A_3 x\left(t - \frac{5}{8}\right) = 0, \tag{38}$$

where

$$A_1 = \begin{bmatrix} -8 & 0 & 1 \\ 1 & -9 & -1 \\ -1 & -1 & -8 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -4 & -3 & -1 \\ 0 & -10 & 2 \\ 0 & -1 & -1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} -7 & -2 & 3 \\ 1 & 9 & -1 \\ 0 & -1 & -6 \end{bmatrix}.$$

Since

$$\mu_\infty(A_1) = \mu_\infty(A_1 + A_2) = -6, \qquad \mu_\infty(A_1 + A_2 + A_3) = -8$$

and

$$\sum_{j=1}^{2} \mu_\infty\left(\sum_{k=1}^{j} A_k\right)(r_{j+1} - r_j) = \mu_\infty(A_1)(r_2 - r_1) + \mu_\infty(A_1 + A_2)(r_3 - r_2)$$
$$= -\frac{9}{4} < -\frac{5}{8}e,$$

the system (38) is nonoscillatory.

Remark 25. Relatively to the scalar case of (1) we would like to remark that Corollaries 20 and 21 are more general than [7, Theorems 10 and 11]. For the scalar case of (2), the same holds to Corollary 23 with respect to [7, Corollary 14].

Comparing the results above with those obtained in [1, Section 2], one easily sees through the examples given here that they are of different kind.

The methods we have followed here are very close to those used in [1, Section 3] to show that (1) and (2) are oscillatory. To see in what measure the results obtained here are in the complement of those obtained in [1], let us recall, for example, that by [1, Corollary 16], if $r(\theta)$ is in D_i^+, $(\mu \circ (-\nu_0))(\theta) \leqslant 0$ and $(\mu \circ (-\nu_1))(\theta) \geqslant 0$, for every $\theta \in [-1, 0]$, and

$$\int_{-1}^{0} \big(\mu \circ (-\nu_1)\big)(\theta)\, d \ln r(\theta) < e, \tag{39}$$

then the system (1) is oscillatory. Comparing this situation with the one described in Corollary 20, we would like to notice the following. By property (iii) of the matrix measures, one has

$$-\big(\mu \circ (-\nu_1)\big)(\theta) \leqslant (\mu \circ \nu_1)(\theta),$$

and so, in particular, condition (34) implies $(\mu \circ (-\nu_1))(\theta) \geqslant 0$. But, assuming that (35) is satisfied, one has

$$-\int_{-1}^{0} \bigl(\mu \circ (-\nu_1)\bigr)(\theta)\, d \ln r(\theta) \leqslant \int_{-1}^{0} \frac{(\mu \circ \nu_1)(\theta)}{r(\theta)}\, dr(\theta)$$
$$\leqslant \frac{1}{r(0)} \int_{-1}^{0} (\mu \circ \nu_1)(\theta)\, dr(\theta) \leqslant -e.$$

So the inverse inequality of (39) is verified.

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