Rolling Maps for the Essential Manifold

L. Machado, F. Pina, and F. Silva Leite

Abstract Computer vision problems typically have geometric constraints. When two cameras view a 3D scene from two distinct positions, or a single camera views a 3D scene from two different locations, there are a number of geometric relations between the 3D points and their projections onto the 2D images. These relations lead to constraints between the image points. In particular, the epipolar constraint encodes the relation between correspondences across two images of the same scene. In a calibrated setting, the epipolar constraint is parameterized by essential matrices, which form the Essential Manifold. The reconstruction of a video from several images of a scene can be formulated as an interpolation problem on this manifold. An approach that simplifies the generation of an interpolating curve consists in projecting the problem to a linear manifold where it can be solved easily, and then projecting back the solution on the nonlinear manifold. The projection is realized by rolling the Essential Manifold, without slip and twist, over an affine tangent space. This gives particular relevance to rolling motions in the context of certain computer vision problems. Having this in mind, we derive the kinematic equations for the rolling motions of the Essential Manifold and present explicit solutions when it rolls along geodesics.

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1 Introduction

Computer vision is a challenging topic which is being used in a wide variety of real world applications, such as earth observation, optical character recognition, 3D model building, medical imaging, machine inspection, automotive safety, match move, motion capture, surveillance, fingerprint recognition and biometrics. We refer to Szeliski [13] and references therein for details concerning multiple applications in this area.

The problem of recovering structure and motion from a sequence of images, also known as stereo matching, is a crucial problem in computer vision and continues to be one of the most active research areas with remarkable progress in imaging and computing hardware (see also Ma et al. [10]). The Essential Manifold plays an important role in this area since it encodes the epipolar constraint. The classical problem of reconstructing a scene, or a video, from several images of the scene can be formulated as an interpolation problem on the Essential Manifold. Typically, it is given an ordered set of time-labeled essential matrices, $E_1, \ldots, E_n$ relating n different consecutive camera views (*snapshots*), and the objective is to calculate a continuum of additional virtual views by computing a smooth interpolating curve through the E_i's. According to Hüper and Silva Leite [6], interpolation problems on manifolds can be efficiently solved via rolling techniques. This approach enables to transform a difficult problem on a curved space into an easy problem on a flat space. Therefore, in order to implement an interpolation algorithm on the Essential Manifold it is particularly important to study rolling motions of this manifold over an affine tangent space where classical interpolation methods may then be applied. There are other problems in the area of computer vision where rolling methods have been used successfully. We refer to Caseiro et al. [1] for a novel application of rolling to solve multi-class classification problems on manifolds.

The classical definition of rolling, without slip and without twist, is presented in Sharpe [12] for manifolds embedded in Euclidean spaces, namely $\mathbb{R}^n$ equipped with the Euclidean metric. These rolling motions result from the action of the group of orientation preserving isometries of the ambient space, which is the special Euclidean group SE_n. Although, according to Nash Theorem [11], every finite dimensional Riemannian manifold can be smoothly isometrically embedded in a sufficiently high-dimensional Euclidean space, finding an appropriate embedding is not necessarily an easy task. For that reason, the concept of rolling has been extended to manifolds embedded in a general Riemannian manifold in Hüper et al. [7]. In the present work though, we consider the Essential Manifold embedded in an appropriate Euclidean space, but since elements in this manifold have a matrix representation, we follow the approach in Hüper and Silva Leite [6] and adjust Sharpe's definition so that the matrix structure is not destroyed.

The organization of the paper is the following. In Sect. 2, we introduce the notions of essential matrix, epipolar constraint and Essential Manifold, and describe the Riemannian structure of the Essential Manifold, following the approach given in Helmke et al. [4] and Ma et al. [10]. Section 3 starts with the notion of a rolling map, which describes the rolling motion of a submanifold of a general Riemannian manifold over another submanifold of equal dimension. This is based on the work of Hüper et al. [7]. The main results, dedicated to the rolling motions of the Essential Manifold over the affine tangent space at a point, appear after the general definition of rolling. More specifically, we adjust the general conditions to our particular case, derive the kinematic equations of rolling, and solve those equations explicitly when the rolling curves are geodesics on the manifold. The paper ends with some remarks and directions for further research.

2 The Essential Manifold

2.1 Geometric Formulation

It is well known from computer vision literature that the intrinsic projective geometry between two views of the same scene is independent of the scene structure and only depends on the cameras internal parameters and relative pose (see, for instance, Hartley and Zisserman [3]). In this paper we deal with calibrated cameras, that is, we assume that the camera parameters are known. We also assume that the scene is static and, for simplicity, we admit that the images are taken by two identical pinhole cameras, with focal length equal to one. The two cameras are denoted by C_1 and C_2 and the corresponding images of the scene structure p are denoted by X_1 and X_2, respectively, as shown in Fig. 1.

Each camera is represented by an orthonormal reference frame and can therefore be described as a change of coordinates relatively to an inertial reference frame. Without loss of generality, we can assume that the inertial frame corresponds to one of the two cameras, say C_1, while the other is positioned and oriented according to

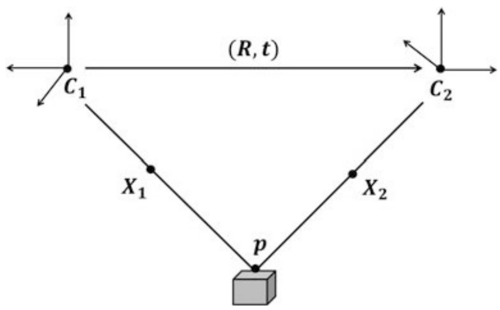

Fig. 1 Geometry between two views of the same scene structure

an element (R, s) of the special Euclidean group $SE_3 = SO_3 \ltimes \mathbb{R}^3$, where R denotes a rotation and s represents a translation vector of the displacement of the first camera C_1 into the second one C_2. Let s_1, s_2 and s_3 be the coordinates of s with respect to the first camera basis ($s = [s_1\ s_2\ s_3]^\top$) and $x_1, x_2 \in \mathbb{R}^3$ be the homogeneous coordinates of the projection of the same point p onto the two image planes of the cameras. If we call $X_1 \in \mathbb{R}^3$ and $X_2 \in \mathbb{R}^3$ the 3D coordinates of the point p relative to the two camera frames, they are related by a rigid body motion:

$$X_2 = RX_1 + s,$$

where $X_i = \lambda_i x_i$, $i = 1, 2$, can be written in terms of the image points x_i, $i = 1, 2$ and the depths λ_i, $i = 1, 2$, ($\lambda_i > 0$). So, the last equation can be written as

$$\lambda_2 x_2 = R\lambda_1 x_1 + s. \tag{1}$$

Consider the isomorphism

$$\begin{aligned} \widehat{(\cdot)} \colon \mathbb{R}^3 &\longrightarrow \mathfrak{so}_3 \\ s = [s_1\ s_2\ s_3]^\top &\longmapsto \hat{s} := \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}, \end{aligned}$$

between $\mathbb{R}^3$ and the Lie algebra of SO_3, which is the set of all 3×3 skew-symmetric matrices, here denoted by $\mathfrak{so}_3$. It is well known and trivial to prove that for any vector $x \in \mathbb{R}^3$, $\hat{s}x = s \times x$ ($\times$ denotes the cross product). Multiplying (on the left) both sides of the Eq. (1) by $\hat{s}$ we then obtain

$$\lambda_2 \hat{s} x_2 = \lambda_1 \hat{s} R x_1.$$

Now, by taking the inner product of both sides of the previous equation with x_2, it follows

$$x_2^\top \hat{s} R x_1 = 0, \tag{2}$$

which is called the *epipolar constraint* (Longuet-Higgins [9]). This intrinsic constraint is independent of depth information and decouples the problem of motion recovery from 3D structure. This problem consists in finding $(R, s) \in SE_3$ using the known image points x_1 and x_2 and the epipolar constraint. The matrix $E = \hat{s}R$ in (2), which captures the relative orientation between the two cameras, is called the *essential matrix* and the set of all such matrices is the so-called Essential Manifold.

2.2 Riemannian Structure of the Normalized Essential Manifold

For many of the applications concerning essential matrices, it is enough to work with a subset of normalized matrices, those of the form $\hat{s}R$, where the translation vector s has norm 1. This set, referred in the literature as the Normalized Essential Manifold, is defined as

$$\mathscr{E} = \left\{ \hat{s}R : \ \hat{s} \in \mathfrak{so}_3, \ R \in \mathrm{SO}_3, \ \frac{1}{2}\mathrm{tr}(\hat{s}^\top \hat{s}) = 1 \right\}.$$

As a consequence of a result in Huang and Faugeras [5], concerning a characterization of essential matrices in terms of their singular values, we can say that all normalized essential matrices have singular values $\{1,1,0\}$. Therefore, using the singular value decomposition, any matrix E in $\mathscr{E}$ can be written as

$$E = UE_0V^\top, \ \text{for some } U, \ V \in \mathrm{SO}_3 \ \text{and } E_0 = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Also, as pointed out in Helmke et al. [4], the Normalized Essential Manifold can also be represented by pairs $(UE_0U^\top, UV^\top)$, where U, V and E_0 are as above. That is, $\mathscr{E} = \mathscr{G}(2,3) \times \mathrm{SO}_3$, where $\mathscr{G}(2,3)$ is the isospectral manifold consisting of the 3×3 real symmetric projection matrices of rank 2 (a Grassmann manifold). From now on we use this parametrization so that $\mathscr{E} = \{(UE_0U^\top, UV^\top) : \quad U, \ V \in \mathrm{SO}_3\}$. We may replace $UV^\top$ by an arbitrary rotation matrix R to obtain the following definition of the Normalized Essential Manifold that will be used throughout the rest of the paper. Also, for the sake of brevity we omit the word normalized and call it simply Essential Manifold.

Definition 1 The *Essential Manifold* is the 5-dimensional smooth manifold defined as

$$\mathscr{E} := \left\{ \left(UE_0U^\top, R\right) : \quad U, \ R \in \mathrm{SO}_3 \right\}, \tag{3}$$

where

$$E_0 = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4}$$

The Essential Manifold can be considered embedded in the Euclidean space $\mathfrak{s}_3 \times \mathbb{R}^{3\times 3}$, where $\mathfrak{s}_3$ denotes the set of all 3×3 real symmetric matrices. The natural metric in this embedding space is defined as

$$\langle (J,K),(L,M) \rangle_{\mathfrak{s}_3 \times \mathbb{R}^{3\times 3}} = \langle J,L \rangle_{\mathfrak{s}_3} + \langle K,M \rangle_{\mathbb{R}^{3\times 3}}, \tag{5}$$

where the metric $\langle .,. \rangle$ in the right hand side is related to the Frobenius norm for matrices, that is, $\langle A, B\rangle = \mathrm{tr}(A^\top B)$. With the above parametrization of the Essential Manifold, the tangent space at a point $P_0 = (\Theta_0 E_0 \Theta_0^\top, R_0) \in \mathscr{E}$ and the corresponding orthogonal space are, respectively, given by:

$$T_{P_0}\mathscr{E} = \left\{\left(\Theta_0\,[\Omega, E_0]\,\Theta_0^\top,\; R_0 C\right) :\; \Omega, C \in \mathfrak{so}_3\right\} \tag{6}$$

or, equivalently,

$$T_{P_0}\mathscr{E} = \left\{\left(\Theta_0 \begin{bmatrix} 0 & \Lambda \\ \Lambda^\top & 0 \end{bmatrix} \Theta_0^\top,\; R_0 C\right) :\; \Lambda \in \mathbb{R}^{2\times 1},\; C \in \mathfrak{so}_3\right\} \tag{7}$$

and

$$(T_{P_0}\mathscr{E})^\perp = \left\{\left(\Theta_0 \begin{bmatrix} B & 0 \\ 0 & b \end{bmatrix} \Theta_0^\top,\; R_0 S\right) :\; B \in \mathfrak{s}_2,\; b \in \mathbb{R},\; S \in \mathfrak{s}_3\right\}. \tag{8}$$

Note that, as expected, the dimensions of the above spaces match with the dimension of the embedding space, which is 15. Indeed, $\dim(T_{P_0}\mathscr{E}) = 5$ and $\dim\left((T_{P_0}\mathscr{E})^\perp\right) = 10$.

3 Rolling the Essential Manifold

We start this section with the important definition of a rolling map for general manifolds and then specialize to the situation when the Essential Manifold rolls, without slip and twist, over the affine tangent space at a point. The kinematic equations for this rolling motion are therefore derived.

3.1 Rolling Maps

We gather the necessary information about rolling maps, so that we can describe the rolling motion of the Essential Manifold later. As already mentioned, the classical definition of a rolling map, for manifolds embedded in Euclidean spaces, appeared first in Sharpe [12]. In the meanwhile, it has been refined and generalized in order to accommodate manifolds embedded in a general Riemannian manifold. We follow closely the notations in Hüper and Silva Leite [6], but include a more general definition contained in Hüper et al. [7]. In this context, a rolling map describes how two connected manifolds M_0 and M_1 of the same dimension n, both isometrically embedded in the same Riemannian complete m-dimensional manifold $\overline{M}$ ($1 \leq n < m$), roll on each other without slipping and twisting. These motions are described by the action of the group of isometries on the embedding manifold $\overline{M}$, which preserve orientations. Let us recall that, if $\overline{M}$ is equipped with the tensor metric g, an isometry on $\overline{M}$ is a diffeomorphism $l : \overline{M} \to \overline{M}$ which preserves g, that is, $l^* g = g$, where l^* denotes the pullback of l. Furthermore, the group of isometries on $\overline{M}$, denoted by $\mathsf{Isom}(\overline{\mathsf{M}})$ is a Lie group, whose dimension is never greater than $m(m+1)/2$ (Kobayashi [8]). The rolling map will be defined as a curve in the connected component of $\mathsf{Isom}(\overline{\mathsf{M}})$ that contains the identity, satisfying several conditions to be presented in Definition 2. This subgroup will be denoted by $\overline{G}$. So, a rolling map on a closed interval $I = [0, \tau] \subset \mathbb{R}$ ($\tau > 0$) can be described using the following pair of mappings:

$$\begin{array}{rl} h : I \to \overline{G} \\ t \mapsto h(t) \end{array} \quad \text{and} \quad \begin{array}{rl} h(t) : \overline{M} \to \overline{M} \\ p \mapsto q = h(t)(p) \end{array}.$$

Let $x \in \overline{M}$ be a point and $\eta \in T_x\overline{M}$ be a tangent vector. This means that there exists a smooth curve $y :]-\varepsilon, \varepsilon[\to \overline{M}$ such that $y(0) = x$ and $\dot{y}(0) = \eta$. We denote by $h(t)_*$ the pushforward (differential) of $h(t)$. Also, from the action of $\overline{G}$ on $\overline{M}$, we may define the following actions, which will be used in the definition of rolling map (Definition 2).

$$\dot{h}(t)(x) := \frac{\mathrm{d}}{\mathrm{d}\sigma} \left[h(\sigma)(x)\right]\Big|_{\sigma=t}, \tag{9}$$

$$\left(\dot{h}(t) \circ h(t)^{-1}\right)(x) := \frac{\mathrm{d}}{\mathrm{d}\sigma} \left[\left(h(\sigma)\, h(t)^{-1}\right)(x)\right]\Big|_{\sigma=t}, \tag{10}$$

$$\left(\dot{h}(t) \circ h(t)^{-1}\right)_* (\eta) := \frac{\mathrm{d}}{\mathrm{d}\sigma} \left[\left(\dot{h}(t) \circ h(t)^{-1}\right)(y(\sigma))\right]\Big|_{\sigma=0}. \tag{11}$$

Definition 2 Let M_0 and M_1 be two n-dimensional connected manifolds isometrically embedded in an m-dimensional complete Riemannian manifold $\overline{M}$ and let $\overline{G}$ be the connected component of the group of isometries of $\overline{M}$ that contains the identity. A *rolling map of* M_1 *over* M_0*, without slipping and twisting*, is a smooth curve $h : I \to \overline{G}$, satisfying, for all $t \in I$, the following three properties:

1. *Rolling conditions*: There exists a smooth curve $\alpha_1 : I \to M_1$, such that

   a. $h(t)(\alpha_1(t)) \in M_0$;
   b. $T_{h(t)(\alpha_1(t))}(h(t)(M_1)) = T_{h(t)(\alpha_1(t))}M_0$.

   The curve α_1 is called the *rolling curve* and the curve $\alpha_0 : I \to M_0$, defined by

   $$\alpha_0(t) = h(t)(\alpha_1(t)), \tag{12}$$

   is called the *development of* α_1 *on* M_0.
2. *No-slip condition*:

   $$\left(\dot{h}(t) \circ h(t)^{-1}\right)(\alpha_0(t)) = 0. \tag{13}$$

3. *No-twist conditions*:

   a. (Tangential part)

$$\left(\dot{h}(t) \circ h(t)^{-1}\right)_* \left(T_{\alpha_0(t)}M_0\right) \subset (T_{\alpha_0(t)}M_0)^{\perp}, \tag{14}$$

   b. (Normal part)

$$\left(\dot{h}(t) \circ h(t)^{-1}\right)_* (T_{\alpha_0(t)}M_0)^{\perp} \subset T_{\alpha_0(t)}M_0. \tag{15}$$

Remark 1

1. Rolling along piecewise-smooth curves only requires a minor adjustment in the conditions, involving derivatives, of the previous definition, replacing "for all t" by "for almost all t".
2. The first rolling condition means that, during the motion, the development curve α_0 is being drawn on M_0 by the point of contact of the moving manifold $h(t)(M_1)$ and the static manifold M_0. The second rolling condition means that, at each time t, both manifolds $h(t)(M_1)$ and M_0 have the same tangent space.
3. The no-slip condition is equivalent to $\dot{\alpha_0}(t) = h(t)_*(\dot{\alpha_1}(t))$. So, this condition has the interpretation that the velocities of the rolling curve and of its development at the point of contact are the same.
4. An interpretation for the no-twist conditions is not so easy to obtain. But Godoy et al. in [2] proved that these conditions can be given an interesting geometric interpretation as follows:

   a. Tangential part: A vector field $Y(t)$ is tangent parallel along the curve $\alpha_1(t)$ if, and only if, $V(t) = h(t)_*Y(t)$ is tangent parallel along $\alpha_0(t)$.
   b. Normal part: A vector field $Z(t)$ is normal parallel along the curve $\alpha_1(t)$ if, and only if, $V(t) = h(t)_*Z(t)$ is normal parallel along $\alpha_0(t)$.

5. In Sharpe [12], it has been proven that given any smooth curve on M_0, there exists a unique rolling map along that curve. This property of existence and uniqueness has been generalized to any Riemannian submanifolds in Hüper et al. [7].

3.2 Rolling Maps for the Essential Manifold

In this section we specialize the rolling maps to the particular situation when M_1 is the Essential Manifold $\mathscr{E}$, and M_0 is the affine tangent space to $\mathscr{E}$ at a particular point P_0, $T^{\text{aff}}_{P_0}\mathscr{E}$. Notice that M_0 and M_1 are assumed to be embedded submanifolds of $\overline{M} = \mathfrak{s}_3 \times \mathbb{R}^{3\times3}$, endowed with the Riemannian metric defined in (5). The approach we take here follows that of Hüper and Silva Leite [6], where the rolling of Grassmann manifolds and of rotation groups has been studied. We recall that, according to our definition of the Essential Manifold given in (3), elements in $\mathscr{E}$

are represented by pairs. We must define the group of isometries $\overline{G}$ of $\overline{M}$. For that, let us start with the Lie group $G = \mathrm{SO}_3 \times \mathrm{SO}_3 \times \mathrm{SO}_3$. It is an easy task to show that it acts transitively on $\mathscr{E}$ via equivalence:

$$\begin{array}{rcl} \overline{\sigma}: \qquad G \times \mathscr{E} & \longrightarrow & \mathscr{E} \\ \Big((U, V, W), (\Theta E_0 \Theta^\top, R)\Big) & \longmapsto & \Big(U \Theta E_0 \Theta^\top U^\top, V R W^\top\Big) \end{array}. \tag{16}$$

Consider now the group $\overline{G} = G \ltimes \left(\mathfrak{s}_3 \times \mathbb{R}^{3\times 3}\right)$, with the product rule

$$\begin{aligned} &\Big(U_1, V_1, W_1, X_1, Y_1\Big)\Big(U_2, V_2, W_2, X_2, Y_2\Big) \\ &\qquad = \Big(U_1 U_2, V_1 V_2, W_1 W_2, U_1 X_2 U_1^\top + X_1, V_1 Y_2 W_1^\top + Y_1\Big), \end{aligned}$$

and inverse

$$\Big(U, V, W, X, Y\Big)^{-1} = \Big(U^\top, V^\top, W^\top, -U^\top X U, -V^\top Y W\Big). \tag{17}$$

The group $\overline{G}$ is connected and acts on $\mathfrak{s}_3 \times \mathbb{R}^{3\times 3}$ via

$$\begin{array}{rcl} \overline{G} \times \left(\mathfrak{s}_3 \times \mathbb{R}^{3\times 3}\right) & \longrightarrow & \mathfrak{s}_3 \times \mathbb{R}^{3\times 3} \\ \Big((U, V, W, X, Y), (A, B)\Big) & \longmapsto & \left(U A U^\top + X, V B W^\top + Y\right) \end{array}. \tag{18}$$

We can conclude that $\overline{G}$ is the isometry group of $\overline{M} = \mathfrak{s}_3 \times \mathbb{R}^{3\times 3}$.

Now, if $P_0 = (\Theta_0 E_0 \Theta_0^\top, R_0)$ is an arbitrary point in $\mathscr{E}$, $\alpha_1 : [0, \tau] \to \mathscr{E}$, defined by $\alpha_1(t) = \left(U(t)\Theta_0 E_0 \Theta_0^\top U(t)^\top, V(t) R_0 W(t)^\top\right)$ is a curve on $\mathscr{E}$ starting from P_0 at $t = 0$. The transitive action of G on $\mathscr{E}$ defined by (16), ensures that any curve on $\mathscr{E}$ has this form. Our goal is to find conditions under which the map

$$\begin{array}{rcl} h : \left[0, \tau\right] & \longrightarrow & \overline{G} \\ t & \longmapsto & h(t) = (U(t)^\top, V(t)^\top, W(t)^\top, X(t), Y(t)) \end{array}, \tag{19}$$

is a rolling map of the Essential Manifold $\mathscr{E}$ over its affine tangent space $T_{P_0}^{\mathrm{aff}}\mathscr{E}$, along

$$\alpha_1(t) = \left(U(t)\Theta_0 E_0 \Theta_0^\top U(t)^\top, V(t) R_0 W(t)^\top\right),$$

with development curve

$$\alpha_0(t) = h(t)(\alpha_1(t)) = \left(\Theta_0 E_0 \Theta_0^\top + X(t), R_0 + Y(t)\right) = P_0 + Z(t) \in M_0, \tag{20}$$

where $Z(t) = (X(t), Y(t)) \in \mathfrak{s}_3 \times \mathbb{R}^{3\times 3}$.

First, we must rewrite (9)–(11) for our particular situation.

Let (A, B) be a point in $\mathfrak{s}_3 \times \mathbb{R}^{3\times 3}$ and $(\xi, \eta) \in \mathfrak{s}_3 \times \mathbb{R}^{3\times 3}$ be a tangent vector to a smooth curve $t \in]-\varepsilon, \varepsilon[\longmapsto y(t) = \big(A(t), B(t)\big) \in \mathfrak{s}_3 \times \mathbb{R}^{3\times 3}$ that satisfies $y(0) = \big(A(0), B(0)\big) = (A, B)$ and $\dot{y}(0) = (\xi, \eta)$. Then, since

$$h(t)(A, B) = \big(U(t)^\top A U(t) + X(t), V(t)^\top B W(t) + Y(t)\big),$$

one gets

$$\begin{aligned}
&\dot{h}(t)((A, B)) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[h(\sigma)((A, B))\right]\Big|_{\sigma=t} \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\big(U(\sigma)^\top A U(\sigma) + X(\sigma), V(\sigma)^\top B W(\sigma) + Y(\sigma)\big)\right]\Big|_{\sigma=t} \\
&\quad = \big(\dot{U}(t)^\top A U(t) + U(t)^\top A \dot{U}(t) + \dot{X}(t), \dot{V}(t)^\top B W(t) + V(t)^\top B \dot{W}(t) + \dot{Y}(t)\big).
\end{aligned} \tag{21}$$

This is the counterpart of (9). Now,

$$\begin{aligned}
&h(\sigma)\, h(t)^{-1} \\
&\quad = \big(U(\sigma)^\top U(t), V(\sigma)^\top V(t), W(\sigma)^\top W(t), -U(\sigma)^\top U(t) X(t) U(t)^\top U(\sigma) + X(\sigma), \\
&\qquad - V(\sigma)^\top V(t) Y(t) W(t)^\top W(\sigma) + Y(\sigma)\big),
\end{aligned} \tag{22}$$

so that, the counterpart of (10) is

$$\begin{aligned}
&\big(\dot{h}(t) \circ h(t)^{-1}\big)((A, B)) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\big(h(\sigma)\, h(t)^{-1}\big)((A, B))\right]\Big|_{\sigma=t} \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}\sigma}\Big|_{\sigma=t}\big[(U^\top(\sigma) U(t) A U(t)^\top U(\sigma) - U^\top(\sigma) U(t) X(t) U^\top(t) U(\sigma) + X(\sigma), \\
&\qquad V^\top(\sigma) V(t) B W(t)^\top W(\sigma) - V(\sigma)^\top V(t) Y(t) W(t)^\top W(\sigma) + Y(\sigma))\big] \\
&\quad = \big(\dot{U}(t)^\top U(t) A + A U(t)^\top \dot{U}(t) - \dot{U}(t)^\top U(t) X(t) - X(t) U(t)^\top \dot{U}(t) + \dot{X}(t), \\
&\qquad \dot{V}(t)^\top V(t) B + B W(t)^\top \dot{W}(t) - \dot{V}(t)^\top V(t) Y(t) - Y(t) W(t)^\top \dot{W}(t) + \dot{Y}(t)\big).
\end{aligned} \tag{23}$$

Finally, the counterpart of (11) is written as

$$\begin{aligned}
&\left(\dot{h}(t)\circ h(t)^{-1}\right)_*((\xi,\eta))\\
&= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\left(\dot{h}(t)\circ h(t)^{-1}\right)(A(\sigma),B(\sigma))\right]\Big|_{\sigma=0} \qquad (24)\\
&= \left(\dot{U}(t)^\top U(t)\xi + \xi U(t)^\top \dot{U}(t), \dot{V}(t)^\top V(t)\eta + \eta W(t)^\top \dot{W}(t)\right).
\end{aligned}$$

3.3 The Kinematic Equations of Rolling

In this section we derive the kinematic equations for the rolling motion by imposing the no-slip and no-twist conditions on $h(t)$ given by (19). Taking into account (23) and the expression for α_0 given by (20), the no-slip condition (13) can be rewritten as

$$\begin{cases} \dot{U}(t)^\top U(t)\Theta_0 E_0 \Theta_0^\top + \Theta_0 E_0 \Theta_0^\top U(t)^\top \dot{U}(t) + \dot{X}(t) = 0 \\ \dot{V}(t)^\top V(t) R_0 + R_0 W(t)^\top \dot{W}(t) + \dot{Y}(t) = 0 \end{cases}. \qquad (25)$$

If we define the skew-symmetric matrices Ω_U, Ω_V and Ω_W by

$$\Omega_U := \Theta_0^\top \dot{U}^\top U \Theta_0, \quad \Omega_V := R_0^\top \dot{V}^\top V R_0, \quad \Omega_W := R_0 \dot{W}^\top W R_0^\top, \qquad (26)$$

the no-slip condition takes the form

$$\begin{cases} \dot{X}(t) = -\Theta_0\left[\Omega_U(t), E_0\right]\Theta_0^\top \\ \dot{Y}(t) = \Omega_W(t) R_0 - R_0 \Omega_V(t) \end{cases}. \qquad (27)$$

Now, using (24), the tangential part of the no-twist conditions is equivalent to showing that, for all $(\xi,\eta) \in T_{\alpha_0(t)}M_0$,

$$\left(\dot{U}^\top U\xi + \xi U^\top \dot{U}, \dot{V}^\top V\eta + \eta W^\top \dot{W}\right) \in (T_{\alpha_0(t)}M_0)^\perp. \qquad (28)$$

But, $T_{\alpha_0(t)}M_0 = T_{P_0}\mathscr{E}$ (and similarly for the normal space). So, taking into account the notations (26), the tangential part of the no-twist conditions (28) is equivalent to

$$\left(\left[\Theta_0 \Omega_U \Theta_0^\top, \xi\right], R_0 \Omega_V R_0^\top \eta - \eta R_0^\top \Omega_W R_0\right) \in (T_{P_0}\mathscr{E})^\perp, \qquad (29)$$

for all $(\xi,\eta) \in T_{P_0}\mathscr{E}$. But, according to (6), for $(\xi,\eta) \in T_{P_0}\mathscr{E}$, we have

$$\xi = \Theta_0 \begin{bmatrix} 0 & \Lambda \\ \Lambda^\top & 0 \end{bmatrix} \Theta_0^\top, \quad \Lambda \in \mathbb{R}^{2\times 1} \quad \text{and} \quad \eta = R_0 C, \quad C \in \mathfrak{so}_3. \qquad (30)$$

Hence, writing the skew-symmetric matrix Ω_U as

$$\Omega_U = \begin{bmatrix} \Omega_1 & \Omega_2 \\ -\Omega_2^\top & 0 \end{bmatrix},$$

where $\Omega_1 \in \mathfrak{so}_2$, $\Omega_2 \in \mathbb{R}^{2\times 1}$, and taking into account that

$$\left[\Theta_0 \Omega_U \Theta_0^\top, \xi\right] = \Theta_0 \begin{bmatrix} \Omega_2 \Lambda^\top + \Lambda \Omega_2^\top & \Omega_1 \Lambda \\ -\Lambda^\top \Omega_1 & -2\Omega_2^\top \Lambda \end{bmatrix} \Theta_0^\top,$$

the characterization of the orthogonal space (8) enables us to conclude that

$$\Omega_1 \Lambda = 0, \quad \text{for all} \quad \Lambda \in \mathbb{R}^{2\times 1}.$$

This implies that $\Omega_1 = 0$ and, therefore, Ω_U must have the constrained structure

$$\Omega_U = \begin{bmatrix} 0 & \Omega_2 \\ -\Omega_2^\top & 0 \end{bmatrix}. \tag{31}$$

Additionally, when $\eta = R_0 C$, $C \in \mathfrak{so}_3$, the second component in (29) should be of the form $R_0 S$, with $S \in \mathfrak{s}_3$. This requires that the matrix $\left(\Omega_V C - C R_0^\top \Omega_W R_0\right)$ is symmetric, for all $C \in \mathfrak{so}_3$. Using this requirement, and after some simple calculations, one concludes that this is equivalent to

$$\left[\Omega_V + R_0^\top \Omega_W R_0, C\right] = 0, \quad \text{for all} \quad C \in \mathfrak{so}_3.$$

Hence, $\Omega_V + R_0^\top \Omega_W R_0 = 0$, that is

$$\Omega_V = -R_0^\top \Omega_W R_0. \tag{32}$$

Therefore, the *tangential part of the no-twist conditions* for the Essential Manifold is equivalent to requiring that

$$\Omega_U = \begin{bmatrix} 0 & \Omega_2 \\ -\Omega_2^\top & 0 \end{bmatrix} \qquad \text{and} \qquad \Omega_V = -R_0^\top \Omega_W R_0. \tag{33}$$

Finally, we must impose the normal part of the no-twist conditions, which is equivalent to showing that, for all $(\xi, \eta) \in (T_{\alpha_0(t)} M_0)^\perp$,

$$\left(\dot{U}^\top U \xi + \xi U^\top \dot{U}, \dot{V}^\top V \eta + \eta W^\top \dot{W}\right) \in T_{\alpha_0(t)} M_0. \tag{34}$$

But it turns out that if conditions (33) hold, the normal part of the no-twist conditions holds as well. Indeed, the previous condition is equivalent to

$$\left(\left[\Theta_0 \Omega_U \Theta_0^\top, \xi\right], R_0 \Omega_V R_0^\top \eta - \eta R_0^\top \Omega_W R_0\right) \in T_{\alpha_0(t)} M_0. \tag{35}$$

So, since $(\xi, \eta) \in (T_{\alpha_0(t)}M_0)^{\perp} = (T_{P_0}\mathscr{E})^{\perp}$, we must have

$$\xi = \Theta_0 \begin{bmatrix} B & 0 \\ 0 & b \end{bmatrix} \Theta_0^{\top}, \quad B \in \mathfrak{s}_2,\ b \in \mathbb{R} \qquad \text{and} \qquad \eta = R_0 S, \quad S \in \mathfrak{s}_3. \tag{36}$$

Hence, using (33), after some calculations we obtain that

$$\left[\Theta_0 \Omega_U \Theta_0^{\top}, \xi\right] = \Theta_0 \begin{bmatrix} 0 & \Omega_2 b - B\Omega_2 \\ -\Omega_2^{\top} B + b\Omega_2^{\top} & 0 \end{bmatrix} \Theta_0^{\top},$$

which is in accordance with the characterization of the tangent space (7). Moreover, the second component presented in relation (35) should be of the form $R_0 C$, with $C \in \mathfrak{so}_3$ and, taking into account (33), this requires that the matrix $(\Omega_V S + S\Omega_V)$ must be skew-symmetric, for all $S \in \mathfrak{s}_3$. A few computations show that this requirement is verified. Thus, *the no-twist conditions reduce to Eq. (33)*.

Now, if the second condition in (33) is used in (27), one obtains

$$\begin{cases} \dot{X}(t) = -\Theta_0 \left[\Omega_U(t), E_0\right] \Theta_0^{\top} \\ \dot{Y}(t) = -2R_0 \Omega_V(t) \end{cases}. \tag{37}$$

The no-slip condition reduces to Eq. (37).

We can now state the main theorem.

Theorem 1 *Let* $\Omega_U(t),\ \Omega_V(t) \in \mathfrak{so}_3$ *with* $\Omega_U = \begin{bmatrix} 0 & \Omega_2 \\ -\Omega_2^{\top} & 0 \end{bmatrix}$, $\Omega_2 \in \mathbb{R}^{2\times 1}$. *If* (U, V, W, X, Y) *is the solution of the following system of differential equations, evolving on* $\overline{G}$,

$$\begin{cases} \dot{U}(t) = -U(t)\Theta_0 \Omega_U(t) \Theta_0^{\top} \\ \dot{V}(t) = -V(t) R_0 \Omega_V(t) R_0^{\top} \\ \dot{W}(t) = W(t)\Omega_V(t) \\ \dot{X}(t) = -\Theta_0 \left[\Omega_U(t), E_0\right] \Theta_0^{\top} \\ \dot{Y}(t) = -2R_0 \Omega_V(t) \end{cases}, \tag{38}$$

with initial condition at the identity element of $\overline{G}$, *that is,* $(U(0), V(0), W(0), X(0), Y(0)) = (I, I, I, 0, 0)$, *then*

$$t \mapsto h(t) = (U(t)^{\top}, V(t)^{\top}, W(t)^{\top}, X(t), Y(t)) \in \overline{G}$$

is a rolling map (in the sense of Definition 2) of the Essential Manifold $\mathscr{E}$ over the affine tangent space at the point $P_0 = (\Theta_0 E_0 \Theta_0{}^\top, R_0)$, along the rolling curve

$$t \mapsto \alpha_1(t) = \big(U(t)\Theta_0 E_0 \Theta_0^\top U(t)^\top, V(t) R_0 W(t)^\top\big),$$

with development curve

$$t \mapsto \alpha_0(t) = \big(\Theta_0 E_0 \Theta_0^\top + X(t), R_0 + Y(t)\big).$$

Proof We have already proved, before the statement of the theorem, that Eq. (38) encode the no-slip and the no-twist conditions. Since the curve α_1 clearly lives in the manifold $\mathscr{E}$ and $\alpha_0(t) = h(t)(\alpha_1(t)) = P_0 + Z(t)$, with $Z(t) = (X(t), Y(t))$, to complete the proof it is enough to show that $Z(t) \in T_{P_0}\mathscr{E}$. But since $\Omega_U(t)$ and $\Omega_V(t)$ are skew-symmetric, it follows from the last two equations of (38) that $\dot{Z}(t) \in T_{P_0}\mathscr{E}$. This, together with the initial condition $Z(0) = 0$, implies that $Z(t) \in T_{P_0}\mathscr{E}$, that is, $\alpha_0(t) \in T_{P_0}^{\text{aff}}\mathscr{E}$.

Remark 2 Equations (38), which encode the non-holonomic constraints of no-slip and no-twist are called the **kinematic equations** for rolling the Essential Manifold over the affine tangent space at the point P_0.
The choice of Ω_U and Ω_V completely determine the solutions of the kinematic equations and, consequently, the rolling curve (and its development). For that reason, we say that these two functions are the "control functions" of the motion.

3.4 Rolling Along Geodesics

For the special situation where the control functions are constant, say $\Omega_U(t) = \Omega_U$ and $\Omega_V(t) = \Omega_V$, the solution of the kinematic equations (38), with initial condition $(U(0), V(0), W(0), X(0), Y(0)) = (I, I, I, 0, 0)$, can be solved explicitly and

$$\begin{cases} U(t) = \Theta_0 e^{-t\Omega_U} \Theta_0^\top \\ V(t) = R_0 e^{-t\Omega_V} R_0^\top \\ W(t) = e^{t\Omega_V} \\ X(t) = -t\Theta_0\big[\Omega_U, E_0\big]\Theta_0^\top \\ Y(t) = -2tR_0\Omega_V \end{cases}. \tag{39}$$

In this case, the rolling curve

$$t \mapsto \alpha_1(t) = \big(\Theta_0 e^{-t\Omega_U} E_0 e^{t\Omega_U} \Theta_0^\top, R_0 e^{-2t\Omega_V}\big) \tag{40}$$

is a geodesic on $\mathscr{E}$, passing through P_0 (at $t = 0$) and, consequently,

$$t \mapsto \alpha_0(t) = P_0 + (X(t), Y(t)) = P_0 + t\left(\Theta_0\,[E_0, \Omega_U]\,\Theta_0^\top,\ 2R_0\Omega_V^\top\right) \tag{41}$$

is also a geodesic in the affine tangent space $T_{P_0}^{\text{aff}}\mathscr{E}$, satisfying $\alpha_0(0) = P_0$. The second statement is obvious since a geodesic in the affine space is a straight line. The first statement can be checked differentiating α_1 twice and noticing that $\ddot{\alpha}_1(t)$ belongs to $(T_{\alpha_1(t)}\mathscr{E})^\perp$. Indeed, using the fact that $e^{t\Omega_U}\Omega_U e^{-t\Omega_U} = \Omega_U$ and the anticommutativity of the matrix commutator, we can write

$$\begin{aligned}\dot{\alpha}_1(t) &= \left(\Theta_0 e^{-t\Omega_U}\left[E_0, e^{t\Omega_U}\Omega_U e^{-t\Omega_U}\right] e^{t\Omega_U}\Theta_0^\top,\ R_0 e^{-2t\Omega_V}(-2\Omega_V)\right)\\ &= \left(\Theta_0 e^{-t\Omega_U}\,[E_0, \Omega_U]\, e^{t\Omega_U}\Theta_0^\top,\ R_0 e^{-2t\Omega_V}(-2\Omega_V)\right).\end{aligned} \tag{42}$$

Differentiating again and simplifying, we obtain

$$\ddot{\alpha}_1(t) = \left(\Theta_0 e^{-t\Omega_U}\,[[E_0, \Omega_U], \Omega_U]\, e^{t\Omega_U}\Theta_0^\top,\ R_0 e^{-2t\Omega_V}(4\Omega_V^2)\right). \tag{43}$$

Hence, taking into account that

$$[[E_0, \Omega_U], \Omega_U] = \begin{bmatrix} -2\Omega_2\Omega_2^\top & 0 \\ 0 & 2\Omega_2^\top\Omega_2 \end{bmatrix}, \tag{44}$$

with $-2\Omega_2\Omega_2^\top$ a symmetric matrix and $2\Omega_2^\top\Omega_2$ a real number, we have

$$\ddot{\alpha}_1(t) = \left(\Theta_0 e^{-t\Omega_U}\begin{bmatrix} -2\Omega_2\Omega_2^\top & 0 \\ 0 & 2\Omega_2^\top\Omega_2 \end{bmatrix} e^{t\Omega_U}\Theta_0^\top,\ R_0 e^{-2t\Omega_V}(4\Omega_V^2)\right), \tag{45}$$

which is in accordance with (8). So, $\ddot{\alpha}_1(t)$ belongs to $(T_{\alpha_1(t)}\mathscr{E})^\perp$, that is, the covariant derivative of $\dot{\alpha}_1$ is identically zero and, thus, (40) is a geodesic on $\mathscr{E}$. We summarize the previous in the following corollary of Theorem 1.

Corollary 1 *If the control functions Ω_U and Ω_V are constant skew-symmetric matrices, then*

$$h(t) = \left(\Theta_0 e^{t\Omega_U}\Theta_0^\top,\ R_0 e^{t\Omega_V} R_0^\top,\ e^{-t\Omega_V},\ -t\Theta_0\big[\Omega_U, E_0\big]\Theta_0^\top,\ -2tR_0\Omega_V\right)$$

is the rolling map of the Essential Manifold $\mathscr{E}$ over $T_{P_0}^{aff}\mathscr{E}$, without slipping and twisting, along the geodesic

$$t \mapsto \alpha_1(t) = \left(\Theta_0 e^{-t\Omega_U} E_0 e^{t\Omega_U}\Theta_0^\top, R_0 e^{-2t\Omega_V}\right) \in \mathscr{E}$$

with development curve

$$t \mapsto \alpha_0(t) = P_0 + Z(t) = P_0 + (X(t), Y(t)) = P_0 + t\left(\Theta_0\,[E_0, \Omega_U]\,\Theta_0^\top,\ 2R_0\Omega_V^\top\right),$$

also a geodesic in the affine tangent space $T_{P_0}^{aff}\mathcal{E}$.

4 Final Remarks

We have derived the kinematic equations of rolling the Essential Manifold over the affine tangent space at a point. The Essential Manifold plays a crucial role in 3D computer vision and these rolling motions may be used to efficiently solve interpolation problems on this manifold, by reducing them to simpler interpolation problems on a flat space, the affine space. The kinematic equations of rolling can be seen as control systems evolving on the group of isometries of the embedding space, whose controls are the functions Ω_U and Ω_V. That is, choosing the controls is equivalent to defining the rolling curve. This motivates several questions concerning controllability and optimal control of rolling motions, issues that will be under investigation in the near future.

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