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# Analytical and numerical treatment of oscillatory mixed differential equations with differentiable delays and advances

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ABSTRACT

In this work, we study the oscillatory behaviour of the differential equation of mixed type

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\,\mathrm{d}\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\,\mathrm{d}\eta(\theta)$$

with delays $r(\theta)$ and advances $\tau(\theta)$, both differentiable. Some analytical and numerical criteria are obtained in order to guarantee that all solutions are oscillatory.



## 1. Introduction

The aim of this work is to study the oscillatory behaviour of the differential equation of mixed type

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\mathrm{d}\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\mathrm{d}\eta(\theta) \tag{1}$$

where $x(t) \in \mathbb{R}$, $\nu(\theta)$ and $\eta(\theta)$ are real functions of bounded variation on $[-1, 0]$ normalized so that $\nu(-1) = \eta(-1) = 0$, and $r(\theta)$ and $\tau(\theta)$ are nonnegative real continuous functions on $[-1, 0]$. Taking

$$\|\tau\| = \max\{\tau(\theta) : \theta \in [-1, 0]\},$$

the advance $\tau(\theta)$ will be assumed to satisfy

$$\tau(\theta_0) = \|\tau\| > \tau(\theta), \quad \forall \theta \neq \theta_0. \tag{2}$$

In the case of $\tau(\theta_0) > 0$, the function $\eta(\theta)$ is supposed to be atomic at $\theta_0$, that is, such that

$$\eta(\theta_0^+) - \eta(\theta_0^-) \neq 0. \tag{3}$$

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