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Classifications of solutions of second-order nonlinear neutral differential equations of mixed type

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Abstract
In this paper the authors classified all solutions of the second-order nonlinear neutral differential equations of mixed type,

$$(a(t)(x(t)+bx(t-\tau_1)+cx(t+\tau_2))')'+p(t)x^{\alpha}(t-\sigma_1)+q(t)x^{\beta}(t+\sigma_2)=0,$$

into four classes and obtained conditions for the existence/non-existence of solutions in these classes. Examples are provided to illustrate the main results.
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1 Introduction

This paper is concerned with the second-order nonlinear neutral differential equations of mixed type of the form

$$\big(a(t)\big(x(t)+bx(t-\tau_1)+cx(t+\tau_2)\big)'\big)'+p(t)x^{\alpha}(t-\sigma_1)+q(t)x^{\beta}(t+\sigma_2)=0, \tag{1.1}$$

$t \geq t_0 > 0$, subject to the following conditions:

(c_1) $a \in C^1([t_0,\infty),\mathbb{R})$ and is positive for all $t \geq t_0$;
(c_2) b and c are constants; τ_1, τ_2, σ_1 and σ_2 are nonnegative constants;
(c_3) α and β are the ratio of odd positive integers;
(c_4) $p, q \in C([t_0,\infty),\mathbb{R})$.

By a solution of equation (1.1), we mean a function $x \in C([T_x,\infty),\mathbb{R})$ for some $T_x \geq t_0$, which has the properties $x(t)+bx(t-\tau_1)+cx(t+\tau_2) \in C^1([T_x,\infty),\mathbb{R})$ and $a(t)(x(t)+bx(t-\tau_1)+cx(t+\tau_2)) \in C^1((T_x,\infty),\mathbb{R})$ and satisfies equation (1.1) on $[T_x,\infty)$. As is customary, a solution of equation (1.1) is oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called non-oscillatory. A non-oscillatory solution $x(t)$ of equation (1.1) is said to be weakly oscillatory if $x(t)$ is non-oscillatory and $x'(t)$ is oscillatory for large value of t.

Second-order neutral delay differential equations have applications in problems dealing with vibrating masses attached to an elastic and also appear, as the Euler equation, in some vibrational problems (see [1] and [2]).

In [3] the authors considered equation (1.1) with $q(t) = 0$, $b = h(t)$, $c = 0$ and $\alpha = 1$ for all $t \geq t_0$, and classified all solutions of (1.1) into four classes and obtained criteria for the existence/non-existence of solutions in these classes. In [4] the authors considered equation (1.1) with $q \leq 0$, $b = c(t)$, $c = 0$, $\alpha = 1$, $\beta = 1$ for all $t \geq t_0$, and classified all solutions into four classes and obtained the solutions in these classes. For $p(t) = 0$ or $q(t) = 0$ and $c = 0$ for all $t \geq t_0$, the oscillatory and asymptotic behavior of solutions of equation (1.1) is discussed in [5, 6] and [7].

In [8–12] the authors considered equation (1.1) with $a(t) \equiv 1$, $\alpha = \beta = 1$ or $\alpha = \beta$ and obtained conditions for the oscillation of all solutions of equation (1.1). Motivated by this observation, in this paper we consider the cases $p, q \geq 0$ and p, q changes sign for all large t, to give sufficient conditions in order that every solution of equation (1.1) is either oscillatory or weakly oscillatory and to study the asymptotic nature of non-oscillatory solutions of equation (1.1) with respect to their asymptotic behavior. All the solutions of equation (1.1) may be *a priori* divided into the following classes:

$$M^+ = \{x = x(t) \text{ solution of (1.1): there exists } t_x \geq t_0 \text{ such that } x(t)x'(t) \geq 0, t \geq t_x\},$$

$$M^- = \{x = x(t) \text{ solution of (1.1): there exists } t_x \geq t_0 \text{ such that } x(t)x'(t) \leq 0, t \geq t_x\},$$

$$OS = \{x = x(t) \text{ solution of (1.1): there exists } \{t_n\}, t_n \to \infty \text{ such that } x(t_n) = 0\},$$

$$WOS = \{x = x(t) \text{ solution of (1.1) } x(t) \neq 0 \text{ for all large } t \text{ but } x'(t) \text{ oscillates}\}.$$

In Section 2, we obtain sufficient conditions for the existence/non-existence in the above said classes. In Section 3, we discuss the asymptotic behavior of solutions in the solutions M^+ and M^-. Examples are provided to illustrate the main results.

2 Existence results

First, we examine the existence of solutions of equation (1.1) in the class M^+.

Theorem 2.1 *With respect to the differential equation* (1.1), *assume that*

(H$_1$) $b \geq 0$ *and* $c \geq 0$;
(H$_2$) $p(t) \geq 0$ *for all* $t \geq t_0$;
(H$_3$) $\alpha > \beta$ *and* $\sigma_1 \leq \sigma_2$.

If

$$\limsup_{t\to\infty} \int_{t_0}^{t} \bigl(Lp(s) + q(s)\bigr)\, ds = \infty, \tag{2.1}$$

for every $L > 0$, *then* $M^+ = \emptyset$.

Proof Suppose that equation (1.1) has a solution $x \in M^+$. Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x'(t) \geq 0$ for all $t \geq t_1$. (The proof if $x(t) < 0$ and $x'(t) \leq 0$ is similar for large t.) Let $z(t) = x(t) + bx(t - \tau_1) + cx(t + \tau_2)$ for $t \geq t_1$. Then by assumption (H$_1$) we have $z(t) > 0$ and $z'(t) \geq 0$ for all $t \geq t_1$.

Now,

$$\begin{aligned}\left(\frac{a(t)z'(t)}{x^{\beta}(t-\sigma_2)}\right)' &= \frac{(a(t)z'(t))'}{x^{\beta}(t-\sigma_2)} - \frac{\beta a(t)z'(t)x'(t-\sigma_2)}{x^{\beta+1}(t-\sigma_2)} \\ &\le -p(t)\frac{x^{\alpha}(t-\sigma_1)}{x^{\beta}(t-\sigma_2)} - q(t) \\ &\le -p(t)\frac{x^{\alpha}(t-\sigma_2)}{x^{\beta}(t-\sigma_2)} - q(t) \\ &\le -p(t)x^{\alpha-\beta}(t-\sigma_2) - q(t), \quad t \ge t_1.\end{aligned}$$

Since $x(t) > 0$ and $x'(t) \ge 0$, we have $x(t) \ge L_0 > 0$ and by (H_3), $x^{\alpha-\beta}(t) \ge L > 0$ for all $t \ge t_1 \ge t_2$. Thus

$$\left(\frac{a(t)z'(t)}{x^{\beta}(t-\sigma_2)}\right)' \le -\big(Lp(t) + q(t)\big), \quad t \ge t_2.$$

Integrating the last inequality from t_2 to t, we obtain

$$\frac{a(t)z'(t)}{x^{\beta}(t-\sigma_2)} - \frac{a(t_2)z'(t_2)}{x^{\beta}(t_2-\sigma_2)} \le -\int_{t_2}^{t} \big(Lp(s) + q(s)\big)\, ds.$$

From condition (2.1), we obtain

$$\liminf_{t\to\infty} \frac{a(t)z'(t)}{x^{\beta}(t-\sigma_2)} = -\infty,$$

which contradicts the fact that $z'(t) \ge 0$ for all large t. This completes the proof of the theorem. □

Theorem 2.2 *Assume that condition* (H_1) *holds. Further assume that*

(H_4) $q(t) \ge 0$ *for all* $t \ge t_0$;
(H_5) $\beta > \alpha$.

If

$$\limsup_{t\to\infty} \int_{t_0}^{t} \big(p(s) + L_1 q(s)\big)\, ds = \infty, \tag{2.2}$$

for every $L_1 > 0$, *then* $M^+ = \emptyset$.

Proof Suppose that equation (1.1) has a solution $x \in M^+$. Without loss of generality, we may assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$ and $x'(t) \ge 0$ for all $t \ge t_1$. (The proof is similar if $x(t) < 0$ and $x'(t) \le 0$ for large t.) Let $z(t) = x(t) + bx(t-\tau_1) + cx(t+\tau_2)$ for $t \ge t_1$. Then by assumption (H_1) we have $z(t) > 0$ and $z'(t) \ge 0$ for all $t \ge t_1$. Now,

$$\begin{aligned}\left(\frac{a(t)z'(t)}{x^{\alpha}(t-\sigma_1)}\right)' &= \frac{(a(t)z'(t))'}{x^{\alpha}(t-\sigma_1)} - \frac{\alpha a(t)z'(t)x'(t-\sigma_1)}{x^{\alpha+1}(t-\sigma_1)} \le -p(t) - q(t)\frac{x^{\beta}(t+\sigma_2)}{x^{\alpha}(t-\sigma_1)} \\ &\le -p(t) - q(t)x^{\beta-\alpha}(t-\sigma_1), \quad t \ge t_1.\end{aligned}$$

As in the proof of Theorem 2.1, we have $x^{\beta-\alpha}(t-\sigma_1) \geq L_1 > 0$ for $t \geq t_2 \geq t_1$ by condition (H_5). Thus

$$\left(\frac{a(t)z'(t)}{x^{\alpha}(t-\sigma_1)}\right)' \leq -\bigl(p(t) + L_1 q(t)\bigr), \quad t \geq t_2.$$

Integrating the last inequality from t_2 to t, we have

$$\frac{a(t)z'(t)}{x^{\alpha}(t-\sigma_1)} - \frac{a(t_2)z'(t_2)}{x^{\alpha}(t_2-\sigma_1)} \leq -\int_{t_2}^{t} \bigl(p(s) + L_1 q(s)\bigr)\, ds.$$

From condition (2.2), we obtain

$$\liminf_{t\to\infty} \frac{a(t)z'(t)}{x^{\alpha}(t-\sigma_1)} = -\infty,$$

which contradicts the fact that $z'(t) \geq 0$ for all large t. This completes the proof of the theorem. □

Theorem 2.3 *Assume that conditions* (H_1) *and* (H_4) *hold. Further assume that*

(H_6) $\alpha = \beta$.

If

$$\limsup_{t\to\infty} \int_{t_0}^{t} \bigl(p(s) + q(s)\bigr)\, ds = \infty, \tag{2.3}$$

then $M^+ = \emptyset$.

Proof Suppose that equation (1.1) has a solution $x \in M^+$. Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x'(t) \geq 0$ for all $t \geq t_1$. (The proof is similar if $x(t) < 0$ and $x'(t) \leq 0$ for large t.) Let $z(t) = x(t) + bx(t-\tau_1) + cx(t+\tau_2)$ for $t \geq t_1$. Then by assumption (H_1) we have $z(t) > 0$ and $z'(t) \geq 0$ for all $t \geq t_1$. Now,

$$\begin{aligned}\left(\frac{a(t)z'(t)}{x^{\alpha}(t-\sigma_1)}\right)' &= \frac{(a(t)z'(t))'}{x^{\alpha}(t-\sigma_1)} - \frac{\alpha a(t)z'(t)x'(t-\sigma_1)}{x^{\alpha+1}(t-\sigma_1)} \\ &\leq -p(t) - q(t)\frac{x^{\alpha}(t+\sigma_2)}{x^{\alpha}(t-\sigma_1)} \\ &\leq -\bigl(p(t) + q(t)\bigr), \quad t \geq t_2 \geq t_1.\end{aligned}$$

Integrating the last inequality from t_2 to t, we have

$$\frac{a(t)z'(t)}{x^{\alpha}(t-\sigma_1)} - \frac{a(t_2)z'(t_2)}{x^{\alpha}(t_2-\sigma_1)} \leq -\int_{t_2}^{t} \bigl(p(s) + q(s)\bigr)\, ds.$$

From condition (2.3), we obtain

$$\liminf_{t\to\infty} \frac{a(t)z'(t)}{x^{\alpha}(t-\sigma_1)} = -\infty,$$

which contradicts the fact that $z'(t) \geq 0$ for all large t. This completes the proof of the theorem. □

Theorem 2.4 *Assume that conditions* (H_2)-(H_4) *hold. Further assume that*

(H_7) $-1 < b + c \leq 0$, *with* $b < 0$ *and* $c > 0$.

If

$$\int_{t_0}^{\infty} \frac{1}{a(s)}\,ds = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \bigl(Lp(s) + q(s)\bigr)\,ds = \infty, \tag{2.4}$$

for every $L \geq 0$, *then* $M^+ = \emptyset$.

Proof Suppose that equation (1.1) has a solution $x \in M^+$. Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x'(t) \geq 0$ for all $t \geq t_1$. (The proof if $x(t) < 0$ and $x'(t) \leq 0$ is similar for large t.) Let $z(t) = x(t) + bx(t - \tau_1) + cx(t + \tau_2) \geq (1 + b + c)x(t - \tau_1) > 0$, for $t \geq t_1$. From equation (1.1), we have

$$\begin{aligned}
\bigl(a(t)z'(t)\bigr)' &= -\left(p(t)\frac{x^{\alpha}(t-\sigma_1)}{x^{\beta}(t-\sigma_2)} + q(t)\frac{x^{\beta}(t+\sigma_2)}{x^{\beta}(t-\sigma_2)}\right)x^{\beta}(t-\sigma_2) \\
&\leq -\bigl(Lp(t) + q(t)\bigr)x^{\beta}(t-\sigma_2) \\
&\leq 0, \quad t \geq t_1.
\end{aligned}$$

Hence, $a(t)z'(t)$ is non-increasing for $t \geq t_1$, and we claim that $a(t)z'(t) \geq 0$ for $t \geq t_1$. If $a(t)z'(t) < 0$ for $t \geq t_2$, then $a(t)z'(t) \leq a(t_2)z'(t_2) < 0$ for $t \geq t_2 > 0$. Now,

$$z'(t) \leq \frac{a(t_2)z'(t_2)}{a(t)} < 0$$

and integrating the last inequality from t_2 to t, we obtain

$$z(t) - z(t_2) \leq \int_{t_2}^{t} \frac{a(t_2)z'(t_2)}{a(s)}\,ds.$$

This implies that $z(t) \to -\infty$ as $t \to \infty$, which is a contradiction. Thus $a(t)z'(t) \geq 0$. Now, proceeding as in the proof of Theorem 2.1 and using condition (2.4), we have

$$\lim_{t\to\infty} \frac{a(t)z'(t)}{x^{\beta}(t-\sigma_2)} = -\infty.$$

This contradicts the fact that $z'(t) \geq 0$ for all large t. This completes the proof of the theorem. □

Theorem 2.5 *Assume that conditions* (H_2), (H_4), (H_5) *and* (H_7) *hold. If*

$$\int_{t_0}^{\infty} \frac{1}{a(s)}\,ds = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \bigl(p(s) + L_1 q(s)\bigr)\,ds = \infty, \tag{2.5}$$

for every $L_1 \geq 0$, *then* $M^+ = \emptyset$.

Proof The proof is similar to that of Theorem 2.4 and hence the details are omitted. □

Theorem 2.6 *Assume that conditions* (H$_2$), (H$_4$), (H$_6$) *and* (H$_7$) *hold. If*

$$\int_{t_0}^{\infty} \frac{1}{a(s)}\,ds = \infty \quad \textit{and} \quad \int_{t_0}^{\infty} \bigl(p(s)+q(s)\bigr)\,ds = \infty, \tag{2.6}$$

then $M^+ = \emptyset$.

Proof The proof is similar to that of Theorem 2.4 and hence the details are omitted. □

Next, we examine the problem of the existence of solutions of equation (1.1) in the class M^-.

Theorem 2.7 *Assume that conditions* (H$_1$)-(H$_3$) *and* (H$_7$) *hold. Further assume that*

(H$_8$) $\tau_1 \le \sigma_2$;
(H$_9$) $\int_0^{\gamma} \frac{du}{u^{\beta}} < \infty$ *and* $\int_{-\gamma}^{0} \frac{du}{u^{\beta}} > -\infty$ *for some* $\gamma > 0$.

If

$$\limsup_{t\to\infty} \int_T^t \frac{1}{a(s)} \left(\int_T^s \bigl(Lp(v)+q(v)\bigr)\,dv \right) ds = \infty \tag{2.7}$$

for all $T \ge t_0$ *and for every* $L > 0$, *then* $M^- = \emptyset$.

Proof Suppose that equation (1.1) has a solution $x \in M^-$. Without loss of generality, we may assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$ and $x'(t) \le 0$ for all $t \ge t_1$. (The proof is similar if $x(t) < 0$ and $x'(t) \ge 0$ for large t.) Let $z(t) = x(t) + bx(t-\tau_1) + cx(t+\tau_2)$, then in view of (H$_1$), $z(t) > 0$ and $z'(t) \le 0$ for all $t \ge t_1$. As in the proof of Theorem 2.1, we obtain

$$\frac{a(t)z'(t)}{x^{\beta}(t-\sigma_2)} - \frac{a(t_1)z'(t_1)}{x^{\beta}(t_1-\sigma_2)} \le -\int_{t_1}^{t} \bigl(Lp(s)+q(s)\bigr)\,ds, \quad t \ge t_1$$

or

$$\frac{z'(t)}{x^{\beta}(t-\sigma_2)} \le \frac{-1}{a(t)} \int_{t_1}^{t} \bigl(Lp(s)+q(s)\bigr)\,ds, \quad t \ge t_1. \tag{2.8}$$

Since x is non-increasing and by (H$_8$), we see that $z(t) \le (1+b+c)x(t-\sigma_2) \ge 0$ and

$$z^{\beta}(t) \le (1+b+c)^{\beta} x^{\beta}(t-\sigma_2). \tag{2.9}$$

Combining (2.8) and (2.9), we have

$$\begin{aligned} \frac{z'(t)}{z^{\beta}(t)} &\le \frac{z'(t)}{(1+b+c)^{\beta} x^{\beta}(t-\sigma_2)} \\ &\le \frac{-1}{a(t)(1+b+c)^{\beta}} \int_{t_1}^{t} \bigl(Lp(s)+q(s)\bigr)\,ds, \quad t \ge t_1. \end{aligned}$$

Integrating the last inequality from t_1 to t, yields

$$\int_{z(t_1)}^{z(t)} \frac{ds}{s^\beta} \le -\int_{t_1}^{t} \frac{1}{a(s)(1+b+c)^\beta}\left(\int_{t_1}^{s}\bigl(Lp(v)+q(v)\bigr)\,dv\right)ds$$

or

$$\int_{z(t)}^{z(t_1)} \frac{ds}{s^\beta} \ge \int_{t_1}^{t} \frac{1}{a(s)(1+b+c)^\beta}\left(\int_{t_1}^{s}\bigl(Lp(v)+q(v)\bigr)\,dv\right)ds$$

and by condition (2.7) we see that

$$\limsup_{t\to\infty}\int_{z(t)}^{z(t_1)} \frac{ds}{s^\beta} = \infty, \tag{2.10}$$

which contradicts condition (H_9). This completes the proof of the theorem. □

Theorem 2.8 *Assume that conditions* (H_1), (H_4), (H_5) *and* (H_7) *hold. Further assume that*

(H_{10}) $\tau_1 \le \sigma_1$;
(H_{11}) $\int_0^\gamma \frac{du}{u^\alpha} < \infty$ *and* $\int_{-\gamma}^0 \frac{du}{u^\alpha} > -\infty$, *for some* $\gamma > 0$.

If

$$\limsup_{t\to\infty}\int_{T}^{t} \frac{1}{a(s)}\left(\int_{T}^{s}\bigl(p(v)+L_1q(v)\bigr)\,dv\right)ds = \infty \tag{2.11}$$

for all $T \ge t_0$ *and for every* $L_1 > 0$, *then* $M^- = \emptyset$.

Proof Suppose that equation (1.1) has a solution $x \in M^-$. Without loss of generality, we may assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$ and $x'(t) \le 0$ for all $t \ge t_1$. (The proof is similar if $x(t) < 0$ and $x'(t) \ge 0$ for all large t.) Let $z(t) = x(t) + bx(t-\tau_1) + cx(t+\tau_2)$, then in view of ($H_1$), $z(t) > 0$ and $z'(t) \le 0$ for all $t \ge t_1$. As in the proof of Theorem 2.2, we obtain the inequality

$$\frac{a(t)z'(t)}{x^\alpha(t-\sigma_1)} - \frac{a(t_1)z'(t_1)}{x^\alpha(t_1-\sigma_1)} \le -\int_{t_1}^{t}\bigl(p(s)+L_1q(s)\bigr)\,ds, \quad t \ge t_1$$

or

$$\frac{z'(t)}{x^\alpha(t-\sigma_1)} \le \frac{-1}{a(t)}\int_{t_1}^{t}\bigl(p(s)+L_1q(s)\bigr)\,ds, \quad t \ge t_0. \tag{2.12}$$

Since x is non-increasing and by (H_{10}), we see that $z(t) \le (1+b+c)x(t-\sigma_1) \ge 0$ and then

$$z^\alpha(t) \le (1+b+c)^\alpha x^\alpha(t-\sigma_1). \tag{2.13}$$

Combining (2.12) and (2.13), we have

$$\begin{aligned}\frac{z'(t)}{z^\alpha(t)} &\le \frac{z'(t)}{(1+b+c)^\alpha x^\alpha(t-\sigma_1)}\\ &\le \frac{-1}{a(t)(1+b+c)^\alpha}\int_{t_1}^{t}\bigl(p(s)+L_1q(s)\bigr)\,ds, \quad t \ge t_1.\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.7 and hence the details are omitted. This completes the proof of the theorem. □

Theorem 2.9 *Assume that conditions* (H_1), (H_4), (H_6), (H_7), (H_{10}) *and* (H_{11}) *hold. If*

$$\limsup_{t\to\infty}\int_T^t \frac{1}{a(s)}\left(\int_T^s \bigl(p(v)+q(v)\bigr)\,dv\right)ds=\infty \tag{2.14}$$

for all $T \ge t_0$, *then* $M^- = \emptyset$.

Proof The proof is similar to that of Theorem 2.8 and hence the details are omitted. □

Next, we establish sufficient conditions under which any solution of equation (1.1) is either oscillatory or weakly oscillatory.

Theorem 2.10 *If conditions* (H_1), (H_2), (H_3), (H_4) *and* (2.4) *hold, then every solution of equation* (1.1) *is either oscillatory or weakly oscillatory.*

Proof From Theorem 2.1, it follows that for equation (1.1) we have $M^+ = \emptyset$. To complete the proof, it suffices to show that for equation (1.1), $M^- = \emptyset$. Let x be a solution of equation (1.1) belonging to the class M^-, say that $x(t) > 0$ and $x'(t) \le 0$ for $t \ge t_1 \ge t_0$. (The proof is similar if $x(t) < 0$ and $x'(t) \ge 0$ for all large t.) Let $z(t) = x(t) + bx(t-\tau_1) + cx(t+\tau_2)$; then in view of (H_1), we have $z(t) > 0$ and $z'(t) \le 0$ for all $t \ge t_1$. Proceeding as in the proof of Theorem 2.1, we obtain

$$\frac{a(t)z'(t)}{x^\beta(t-\sigma_2)}-\frac{a(t_1)z'(t_1)}{x^\beta(t_1-\sigma_2)}+\int_{t_1}^t \frac{\beta a(s)z'(s)x'(s-\sigma_2)}{x^{\beta+1}(s-\sigma_2)}\,ds \le -\int_{t_1}^t \bigl(Lp(s)+q(s)\bigr)\,ds.$$

Set

$$w(t)=\frac{a(t)z'(t)}{x^\beta(t-\sigma_2)},\quad t\ge t_1.$$

Then $w(t) < 0$ and for $t \ge t_1$,

$$w'(t)=\frac{(a(t)z'(t))'}{x^\beta(t-\sigma_2)}-\beta\frac{a(t)z'(t)}{x^{\beta+1}(t-\sigma_2)}x'(t-\sigma_2)\le -\bigl(Lp(t)+q(t)\bigr)-\beta w(t)\frac{x'(t-\sigma_2)}{x(t-\sigma_2)}.$$

Integrating the last inequality from t_1 to t and using condition (2.4), we have

$$w(t)\le w(t_1)+\int_{t_1}^t w(s)\left(-\beta\frac{x'(s-\sigma_2)}{x(s-\sigma_2)}\right)ds,\quad t\ge t_2\ge t_1.$$

By Gronwall's inequality, we obtain

$$w(t)\le w(t_1)\left(\frac{x(t_1-\sigma_2)}{x(t-\sigma_2)}\right)^\beta,$$

and so

$$a(t)z'(t)\le w(t_1)x^\beta(t_1-\sigma_2)=M_1<0$$

or

$$z'(t) \leq \frac{M_1}{a(t)}, \quad t \geq t_2.$$

The integration yields

$$z(t) \leq z(t_2) + M_1 \int_{t_2}^{t} \frac{1}{a(s)}\, ds \to -\infty$$

as $t \to \infty$ by condition (2.4). This contradiction completes the proof. □

Theorem 2.11 *If conditions* (H_1), (H_4), (H_5) *and* (2.5) *hold, then every solution of equation* (1.1) *is either oscillatory or weakly oscillatory.*

Proof The proof is similar to that of Theorem 2.10 and hence the details are omitted. □

Theorem 2.12 *If conditions* (H_1), (H_4), (H_6) *and* (2.6) *hold, then every solution of equation* (1.1) *is either oscillatory or weakly oscillatory.*

Proof The proof is similar to that of Theorem 2.10 and hence the details are omitted. □

3 Behavior of solutions in the classes M^+ and M^-

First, we study the asymptotic behavior of solutions in the class M^-.

Theorem 3.1 *Assume that conditions* (H_1)-(H_3) *and* (H_8) *hold. If condition* (2.7) *holds, then for every solution* $x(t) \in M^-$, *we have* $\lim_{t\to\infty} x(t) = 0$.

Proof The argument used in the proof of Theorem 2.7 again leads to (2.10). This implies that $\lim_{t\to\infty} z(t) = 0$. But $z(t) \geq x(t)$ for all $t \geq t_1$ implies that $\lim_{t\to\infty} x(t) = 0$ and the proof is complete. □

Theorem 3.2 *Assume that conditions* (H_1), (H_4), (H_5) *and* (H_{10}) *hold. If condition* (2.11) *holds, then for every solution* $x(t) \in M^-$, *we have* $\lim_{t\to\infty} x(t) = 0$.

Proof The argument used in the proof of Theorem 2.8 again leads to (2.14). This implies that $\lim_{t\to\infty} z(t) = 0$. But $z(t) \geq x(t)$ for all $t \geq t_1$ implies that $\lim_{t\to\infty} x(t) = 0$ and the proof is complete. □

Theorem 3.3 *Assume that conditions* (H_1), (H_4), (H_6) *and* (H_{10}) *hold. If condition* (2.14) *holds, then for every solution* $x(t) \in M^-$, *we have* $\lim_{t\to\infty} x(t) = 0$.

Proof The argument used in the proof of Theorem 2.8 again leads to (2.14). This implies that $\lim_{t\to\infty} z(t) = 0$. But $z(t) \geq x(t)$ for all $t \geq t_1$ implies that $\lim_{t\to\infty} x(t) = 0$ and the proof is complete. □

Finally, we examine the asymptotic behavior of solutions in the class M^+.

Theorem 3.4 *If the assumptions* (H_1)-(H_3) *hold and*

$$\limsup_{t\to\infty}\int_T^t \bigl(Lp(s)+q(s)\bigr)\left(\int_T^s \frac{1}{a(v)}\,dv\right)ds=\infty \tag{3.1}$$

for all $T \ge t_0$, *and any* $L > 0$ *is satisfied, then every solution in the class* M^+ *is unbounded.*

Proof Let x be a solution of equation (1.1) such that $x \in M^+$. Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$ and $x'(t) > 0$ for all $t \ge t_1$, for some $t_1 \ge t_0$. (The proof is similar if $x(t) < 0$ and $x'(t) \le 0$ for all large t.) Let $z(t) = x(t) + bx(t - \tau_1) + cx(t + \tau_2)$; then in view of condition (H_1), $z(t) > 0$ and $z'(t) \ge 0$ for all $t \ge t_1$. Consider the function

$$w(t)=-\frac{a(t)z'(t)}{x^\beta(t-\sigma_2)}\int_{t_1}^t \frac{1}{a(s)}\,ds,\quad t\ge t_2\ge t_1.$$

Then we have for $t \ge t_2$

$$\begin{aligned} w'(t) &= -\frac{z'(t)}{x^\beta(t-\sigma_2)}-\frac{(a(t)z'(t))'}{x^\beta(t-\sigma_2)}\int_{t_1}^t\frac{1}{a(s)}\,ds+\frac{\beta a(t)z'(t)x'(t-\sigma_2)}{x^{\beta+1}(t-\sigma_2)}\int_{t_1}^t\frac{1}{a(s)}\,ds\\ &\ge -\frac{z'(t)}{x^\beta(t-\sigma_2)}+\bigl(Lp(s)+q(s)\bigr)\int_{t_1}^t\frac{1}{a(s)}\,ds. \end{aligned}$$

Integrating the last inequality, we obtain

$$w(t)\ge w(t_2)+\int_{t_2}^t\bigl(Lp(s)+q(s)\bigr)\left(\int_{t_1}^s\frac{1}{a(v)}\,dv\right)ds-\int_{t_2}^t\frac{z'(s)}{x^\beta(s-\sigma_2)}\,ds. \tag{3.2}$$

As the function $\frac{z'(t)}{x^\beta(t-\sigma_2)}$ is positive for $t > t_2$, then the limit $\lim_{t\to\infty}\int_{t_2}^t \frac{z'(s)}{x^\beta(s-\sigma_2)}\,ds$ exists. We claim that it is ∞. Assume that

$$\lim_{t\to\infty}\int_{t_2}^t\frac{z'(s)}{x^\beta(s-\sigma_2)}\,ds=M_4<\infty.$$

In view of (3.1) and (3.2), we have $\limsup_{t\to\infty} w(t) = \infty$, which contradicts $w(t)$ being negative for all large values of t. Thus

$$\lim_{t\to\infty}\int_{t_2}^t\frac{z'(s)}{x^\beta(s-\sigma_2)}\,ds=\infty. \tag{3.3}$$

Now, for all values of $t \ge t_2$, we have $x^\beta(t-\sigma_2) \ge x^\beta(t_1-\sigma_2)$, or $\frac{1}{x^\beta(t-\sigma_2)} \le \frac{1}{x^\beta(t_1-\sigma_2)} = M_5$, and consequently

$$\begin{aligned} \int_{t_2}^t\frac{z'(s)}{x^\beta(t-\sigma_2)}\,ds &\le M_5\int_{t_2}^t z'(s)\,ds\\ &= M_5\bigl(z(t)-z(t_2)\bigr). \end{aligned}$$

From (3.3) we obtain

$$\lim_{t\to\infty} z(t)=\infty. \tag{3.4}$$

Since $z(t) = x(t) + bx(t - \tau_1) + cx(t + \tau_2)$ and $x(t)$ is nondecreasing, we have $z(t) \leq (1 + b + c)x(t + \tau_2)$. In view of (3.4) we get $\lim_{t\to\infty} x(t) = \infty$. This completes the proof. □

Theorem 3.5 *If the assumptions* (H_1), (H_4) *and* (H_5) *hold and*

$$\limsup_{t\to\infty} \int_T^t \bigl(p(s) + L_1 q(s)\bigr)\left(\int_T^s \frac{1}{a(v)}\,dv\right) ds = \infty \tag{3.5}$$

for all $T \geq t_0$, *and for any* $L_1 > 0$ *is satisfied, then every solution in the class* M^+ *is unbounded.*

Proof The proof is similar to that of Theorem 3.4 and hence the details are omitted. □

Theorem 3.6 *If the assumptions* (H_1), (H_4) *and* (H_6) *hold and*

$$\limsup_{t\to\infty} \int_T^t \bigl(p(s) + q(s)\bigr)\left(\int_T^s \frac{1}{a(v)}\,dv\right) ds = \infty \tag{3.6}$$

for all $T \geq t_0$ *is satisfied, then every solution in the class* M^+ *is unbounded.*

Proof The proof is similar to that of Theorem 3.4 and hence the details are omitted. □

4 Examples

In this section we present some examples to illustrate the main results.

Example 1 Consider the differential equation

$$\left(\frac{1}{t}\bigl(x(t) + 2x(t-1) + 3x(t+1)\bigr)'\right)' + \frac{1}{t^2(t-1)^3}x^3(t-1) + \frac{5}{t^2(t+1)}x(t+1) = 0 \tag{4.1}$$

for $t \geq 2$. All the conditions of Theorem 2.1 are satisfied except condition (2.1). We see that equation (4.1) has a solution $x(t) = t \in M^+$ since it satisfies equation (4.1).

Example 2 Consider the differential equation

$$\left(\frac{1}{t}\bigl(x(t) + 4x(t-2) + 5x(t+1)\bigr)'\right)' + \frac{4}{t^2(t-1)}x(t-1) + \frac{6}{t^2(t+2)^3}x^3(t+2) = 0 \tag{4.2}$$

for $t \geq 2$. All the conditions of Theorem 2.2 are satisfied except condition (2.2). We see that equation (4.2) has a solution $x(t) = t \in M^+$ since it satisfies equation (4.2).

Example 3 Consider the differential equation

$$\left(\frac{1}{t}\bigl(x(t) + x(t-1) + x(t+1)\bigr)'\right)' + \frac{1}{t^2(t-1)^3}x^3(t-1) + \frac{2}{t^2(t+1)^3}x^3(t+1) = 0 \tag{4.3}$$

for $t \geq 2$. All the conditions of Theorem 2.3 are satisfied except condition (2.3). We see that equation (4.3) has a solution $x(t) = t \in M^+$ since it satisfies equation (4.3).

Example 4 Consider the differential equation

$$\left(t^2\left(x(t) - 3x(t-1) + x(t+1)\right)'\right)' + \frac{t}{(t-1)}x(t-1) + \frac{t}{(t+2)^3}x^3(t+2) = 0 \tag{4.4}$$

for $t \geq 2$. All the conditions of Theorem 2.5 are satisfied except conditions (H_7) and (2.5). In fact, it has a solution $x(t) = t \in M^+$.

Example 5 Consider the differential equation

$$\left(e^{2t}\left(x(t) + \frac{1}{e}x(t-1) + ex(t+1)\right)'\right)' + e^{4t-6}x^3(t-2) + 2e^{2t+3}x(t+3) = 0 \tag{4.5}$$

for $t \geq 0$. Since the function $x(t) = e^{-t}$ is a solution of equation (4.5), we have $M^- \neq \emptyset$. Moreover, condition (2.7) holds in Theorem 2.7, while condition (H_9) is not satisfied.

Example 6 Consider the differential equation

$$\left(e^{2t}\left(x(t) + \frac{1}{e}x(t-1) + e^2x(t+2)\right)'\right) + e^{\frac{2}{3}(2t-1)}x^{\frac{1}{3}}(t-2) + 2e^{2t+3}x(t+3) = 0 \tag{4.6}$$

for $t \geq 0$. Since the function $x(t) = e^{-t}$ is a solution of equation (4.6), we have $M^- \neq \emptyset$. For this equation condition (2.11) does not hold in Theorem 2.8, while condition (H_{10}) is satisfied.

Example 7 Consider the differential equation (4.5). It is easy to see that all the conditions of Theorem 3.2 are satisfied. In fact, $x(t) = e^{-t} \in M^-$ is a solution of equation (4.5) such that $x(t) = e^{-t} \to 0$ as $t \to \infty$.

Example 8 Consider the differential equation (4.3). It can easily be seen that all the conditions of Theorem 3.4 are satisfied. Here $x(t) = t \in M^+$ is a solution of equation (4.3) such that $x(t) \to \infty$ as $t \to \infty$.

We conclude this paper with the following remark.

Remark 1 In this paper we obtained conditions for the non-existence of solutions in the classes M^+ and M^- and the existence of solutions in the classes OS and WOS. It would be interesting to extend the results of this paper to the following equation:

$$\left(a(t)\left(x(t) + b(t)x(t-\tau_1) + c(t)x(t+\tau_2)\right)'\right)' + p(t)x^{\alpha}(t-\sigma_1) + q(t)x^{\beta}(t+\sigma_2) = r(t),$$

where $r(t)$ is a real valued continuous function.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

ET framed the problem and SP solved the problem. SP modified and made changes in the proof of the theorems. All authors read and approved the manuscript.

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