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BLOCK EIGENVECTORS OBTAINED BY BLOCK HOTELLING DEFLATION

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Abstract

The computation of block eigenvalues and block eigenvectors of a matrix partitioned into blocks is dealt with. A block version of a Hotelling deflation process and a block spectral decomposition are presented.

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1. Introduction

In this note, we present some contribution to the so called *Block Eigenvalue Problem* [1]. Matrices partitioned into blocks have been dealt with in several texts and papers, namely, Egérvary [3], Gellai [4], Martins et al. [6, 7], Pereira [8, 9], Pereira and Vitória [10], Rózsa [11, 12], Vitória [13-15] and Woigt [17].

Block eigenvectors of matrices partitioned into blocks, seemingly, did not deserve the same attention paid to block eigenvalues. Some work in this direction is to be presented in this paper.

The plan of this paper is as follows: in Section 2, we expose the block spectral decomposition of a matrix partitioned into blocks; in Section 3, we present the block version of the Hotelling deflation process, where the left and right block eigenelements are considered.

2. Block Spectral Decomposition

Let A be a block matrix of order mn. If

$$Y_1 A = \Gamma Y_1, \tag{1}$$

where Γ is a block (a matrix of order n) and the block vector Y_1 (a matrix of dimension $n \times mn$) is of full rank, then we say that Γ is a *left block eigenvalue* of A and Y_1 is the corresponding *left block eigenvector*.

In a similar way for

$$AX_1 = X_1 \Lambda, \tag{2}$$

we define the *right block eigenvalue* Λ of A and the corresponding *right block eigenvector* X_1 being of full rank and with dimension $mn \times n$.

In the present paper, we investigate relationships between the right and left block eigenvalues of a given matrix. We state that they are the same.

Lemma 2.1. *A right block eigenvalue is also a left block eigenvalue and reciprocally.*

Proof. For this sake, let us consider the right block eigenvalue problem (2). From $AX_1 = X_1 \Lambda$, we have $(AX_1)^T = (X_1 \Lambda)^T$, i.e., $X_1^T A^T = \Lambda^T X_1^T$.

Bearing in mind that a square matrix is similar to its transpose, we write $\Lambda^T = S^{-1}\Lambda S$ and $A^T = PAP^{-1}$ for S and P nonsingular matrices.

Then, we have: $X_1^T A^T = \Lambda^T X_1^T \Leftrightarrow X_1^T A^T = S^{-1}\Lambda S X_1^T \Leftrightarrow SX_1^T A^T = \Lambda S X_1^T$ $\Leftrightarrow SX_1^T PAP^{-1} = \Lambda S X_1^T \Leftrightarrow SX_1^T PA = \Lambda S X_1^T P.$

By putting $Y_1 = SX_1^T P,$ which is a full rank matrix, we have $Y_1 A = \Lambda Y_1,$ hence Λ is a left block eigenvalue of matrix A.

Mutatis mutandis, we prove that a left block eigenvalue is also a right block eigenvalue. □

So, from now on, we omit the words left and right and use only the expression block eigenvalues.

Block eigenvalues are useful because they have all the spectral informations of the block matrix [9, 10]. Suppose that $\Lambda_1, \Lambda_2, ..., \Lambda_m$ are block eigenvalues of A. Let

$$A = X \operatorname{diag}(\Lambda_1 \Lambda_2 \cdots \Lambda_m) X^{-1}, \tag{3}$$

where X of order mn is nonsingular and $\operatorname{diag}(\Lambda_1 \Lambda_2 \cdots \Lambda_m)$ is the diagonal matrix of the m block eigenvalues. Thus $\operatorname{diag}(\Lambda_1 \Lambda_2 \cdots \Lambda_m)$ is similar to A and we say that $\Lambda_1, \Lambda_2, ..., \Lambda_m$ are a *complete set* of block eigenvalues of A.

Furthermore, $X = [X_1 \quad X_2 \quad \cdots \quad X_m],$ where $X_1, X_2, ..., X_m$ are the right block eigenvectors of A associated to the block eigenvalues $\Lambda_1, \Lambda_2, ..., \Lambda_m,$ so, (3) can be written as

$$A = [X_1 \quad X_2 \quad \cdots \quad X_m] \begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_m \end{bmatrix} [X_1 \quad X_2 \quad \cdots \quad X_m]^{-1}. \tag{4}$$

In a similar way, we have, concerning the left block eigenelements:

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1m} \\ Y_{21} & Y_{22} & \cdots & Y_{2m} \\ \vdots & \cdots & \cdots & \vdots \\ Y_{m1} & Y_{m2} & \cdots & Y_{mm} \end{bmatrix}, \tag{5}$$

where $Y_1, Y_2, ..., Y_m$ are the left block eigenvectors of A associated to the block eigenvalues $\Lambda_1, \Lambda_2, ..., \Lambda_m$ and

$$A = Y^{-1}\operatorname{diag}(\Lambda_1\Lambda_2 \cdots \Lambda_m)Y, \tag{6}$$

where Y of order mn is nonsingular and $\operatorname{diag}(\Lambda_1\Lambda_2 \cdots \Lambda_m)$ is the diagonal matrix of the m block eigenvalues.

So, in the following result, a block version of the spectral decomposition ([5, p. 154]) is presented, that is, a matrix partitioned into blocks, A, is expressed as a sum of matrices, each one is defined in terms of its block eigenvalues and the corresponding right and left block eigenvectors.

Proposition 2.1. *Let A be a square matrix, of order mn, partitioned into $m \times m$ blocks of order n. If $\Lambda_1, \Lambda_2, ..., \Lambda_m$ are a complete set of block eigenvalues of matrix A, corresponding to the right block eigenvectors $X_1, X_2, ..., X_m$, then there are m left block eigenvectors $Y_1, Y_2, ..., Y_m$, corresponding also to the same block eigenvalues $\Lambda_1, \Lambda_2, ..., \Lambda_m$, satisfying the relations*:

(i)

$$Y_i X_k = \delta_{ik} I_n, \quad 1 \leq i, k \leq m.$$

Furthermore, we have

(ii)

$$A = \sum_{i=1}^{m} X_i \Lambda_i Y_i.$$

Proof. (i) From relation (4), we have $AX = XD$, where

$$X = [X_1 \quad X_2 \quad \cdots \quad X_m] = \begin{bmatrix} X_{11} & X_{21} & \cdots & X_{m1} \\ X_{12} & X_{22} & \cdots & X_{m2} \\ \vdots & \cdots & \cdots & \vdots \\ X_{1m} & X_{2m} & \cdots & X_{mm} \end{bmatrix}$$

and

$$D = \operatorname{diag}(\Lambda_1\Lambda_2 \cdots \Lambda_m).$$

We now consider the block matrix

$$Y = X^{-1}, \tag{7}$$

where Y is written as in (5).

Hence multiplying each member of relation $AX = XD$, at right and at left by Y, we get $YAXY = YXDY \Leftrightarrow YA = DY \Leftrightarrow Y_iA = \Lambda_i Y_i$, $i = 1, 2, ..., m$. Thus $Y_1, Y_2, ..., Y_m$ are left block eigenvectors associated to the block eigenvalues $\Lambda_1, \Lambda_2, ..., \Lambda_m$.

From the relation (7), we get

$$YX = I_{mn} \Leftrightarrow \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix} [X_1 \quad X_2 \quad \cdots \quad X_m] = \begin{bmatrix} I_n & 0 & \cdots & 0 \\ 0 & I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_n \end{bmatrix}, \tag{8}$$

where $Y_iX_i = I_n$ and $Y_kX_i = 0_n$, $k \neq i$; $i, k = 1, ..., m$.

(ii) From $A = XDY$, it follows that

$$A = XDY = \begin{bmatrix} X_{11} & X_{21} & \cdots & X_{m1} \\ X_{12} & X_{22} & \cdots & X_{m2} \\ \vdots & \cdots & \cdots & \vdots \\ X_{1m} & X_{2m} & \cdots & X_{mm} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Lambda_m \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1m} \\ Y_{21} & Y_{22} & \cdots & Y_{2m} \\ \vdots & \cdots & \cdots & \vdots \\ Y_{m1} & Y_{m2} & \cdots & Y_{mm} \end{bmatrix}$$

$$= [X_1 \quad X_2 \quad \cdots \quad X_m] \begin{bmatrix} \Lambda_1 Y_1 \\ \Lambda_2 Y_2 \\ \vdots \\ \Lambda_m Y_m \end{bmatrix} = \sum_{i=1}^{m} X_i \Lambda_i Y_i. \qquad \square$$

3. Block Hotelling Deflation

In this section, a block version of the deflation Hotelling process [2, 16] is presented for matrices partitioned into blocks.

Proposition 3.1. *If*

(i) $A \in \mathbb{C}^{mn \times mn}$ *is a block matrix.*

(ii) $\Lambda_1, \Lambda_2, \ldots, \Lambda_m \in \mathbb{C}^{n\times n}$ *are a complete set of block eigenvalues associated*, *respectively*, *to the right block eigenvectors* $X_1, X_2, \ldots, X_m \in \mathbb{C}^{mn\times n}$ *of the matrix A*.

(iii) $Y_1, Y_2, \ldots, Y_m \in \mathbb{C}^{n\times mn}$ *are left block eigenvectors of the matrix A associated*, *respectively*, *to the block eigenvalues* $\Lambda_1, \Lambda_2, \ldots, \Lambda_m \in \mathbb{C}^{n\times n}$ *with* $Y_i X_k = \delta_{ik} I_n,\ 1 \le i,\ k \le m.$

(iv) *For a* $\Lambda_j \neq 0_n,\ 1 \le j \le m,\ B \in \mathbb{C}^{mn\times mn}$ *is a block matrix defined by*

$$B = A - X_j \Lambda_j Y_j. \tag{9}$$

Then

(a) 0_n *is a block eigenvalue of B associated to the block eigenvectors* X_j *and* Y_j;

(b) $\Lambda_1, \ldots, \Lambda_{j-1}, \Lambda_{j+1}, \ldots, \Lambda_m$ *are block eigenvalues of B associated*, *respectively*, *to the block eigenvectors* $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_m$ *and* $Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_m$.

Proof. For simplicity, we consider $j = 1$.

We multiply the two members of the equation (9) on the right by X_k, for $k = 1, \ldots, m$, and using (ii), we obtain

$$BX_k = AX_k - X_1\Lambda_1 Y_1 X_k = X_k\Lambda_k - X_1\Lambda_1(Y_1 X_k). \tag{10}$$

Now, multiplying both members of the equation (9) on the left by Y_k, for $k = 1, \ldots, m$, and using (iii), we have

$$Y_k B = Y_k A - Y_k X_1 \Lambda_1 Y_1 = \Lambda_k Y_k - (Y_k X_1)\Lambda_1 Y_1. \tag{11}$$

We consider the two cases: $k = 1$ and $k \neq 1$.

(a) If $k = 1$, then we have by (iii), $Y_1 X_k = I_n$ and $Y_k X_1 = I_n$. From the equation (10) and using (ii) and (iii), we obtain

$$\begin{aligned} BX_1 &= AX_1 - X_1\Lambda_1Y_1X_1 \\ &= X_1\Lambda_1 - X_1\Lambda_1(Y_1X_1) \\ &= X_1\Lambda_1 - X_1\Lambda_1I_n \\ &= X_1(\Lambda_1 - \Lambda_1) \\ &= X_10_n, \end{aligned}$$

and from the equation (11) and using (iii), we get

$$\begin{aligned} Y_1B &= Y_1A - Y_1X_1\Lambda_1Y_1 \\ &= \Lambda_1Y_1 - (Y_1X_1)\Lambda_1Y_1 \\ &= \Lambda_1Y_1 - I_n\Lambda_1Y_1 \\ &= (\Lambda_1 - \Lambda_1)Y_1 \\ &= 0_nY_1. \end{aligned}$$

Thus, we have that 0_n is a block eigenvalue of B associated to the block eigenvectors X_1 and Y_1.

(b) If $k \neq 1$, then we have by (iii), $Y_1X_k = 0_n$ and $Y_kX_1 = 0_n$. We take the equation (10) and use (ii) and (iii), thus we have

$$BX_k = AX_k - X_1\Lambda_1Y_1X_k = X_k\Lambda_k - X_1\Lambda_1(Y_1X_k) = X_k\Lambda_k$$

and from the equation (11) and using (iii), we obtain

$$Y_kB = Y_kA - Y_kX_1\Lambda_1Y_1 = \Lambda_kY_k - (Y_kX_1)\Lambda_1Y_1 = \Lambda_kY_k.$$

Hence, we have that Λ_k, $k = 2, ..., m$, are block eigenvalues of the matrix B associated, respectively, to the same right block eigenvectors and left block eigenvectors of the matrix A. □

Remark 3.1. For the effective computation of a complete set of block eigenvalues of a given block partitioned matrix, we have to use, successively, Proposition 3.1. Indeed, in each step of the block Hotelling deflation process, it is obtained one block eigenvalue associated to a left block eigenvector and to a right block eigenvector.

4. Concluding Remarks

Scalar spectral decomposition is well studied and an application to differential equations is proposed in [5].

Block spectral decomposition leads us to a different path on the study of block deflation when comparison is done with [10].

Studying the use of block spectral decomposition in matrix differential (and difference) equations, may give some insight on the advantages of the block Hotelling deflation process.

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