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3RD YEAR PUPILS' PROCEDURES TO SOLVE MULTIPLICATION TASKS

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INTRODUCTION

This paper reports on the analysis of the procedures used by third grade pupils who participated in a teaching experiment focused on multiplication. The design and development of the teaching experiment were anchored on two main constructs. The first one, number sense, that was conceived according to the meaning given by McIntosh, Reys & Reys (1992) and has contextualised the work perspective developed according to what is recommended in curricular reference documents such as the Standards (NCTM, 2000) and the Portuguese Sylabus. The second one relates to the construction of learning trajectories. The design of the multiplication tasks presented to the pupils, as well as their sequence and planning of how they could be used in the classroom, were performed considering that a hypothetical learning trajectory underlines "the importance of having a goal and rationale for teaching decisions and the hypothetical nature of such thinking" (Simon, 1995, p. 136) and that its development implies a focused attention on the pupils' learning. The analysis of pupils' solutions provides information on their mathematical ideas (Stein, Remillard & Smith, 2007). Describing their procedures and realize their evolution enables us to make decisions on the adjustments to be made to the hypothetical learning trajectory.

THEORETICAL FRAMEWORK

Research literature focused on multiplication is quite extensive. We can identify an emphasis on prominent lines: semantic types of situations (Greer, 1992), intuitive models (Mulligan & Mitchelmore, 1997) and computational strategies. The last one, more related with the focus of this paper, includes empirical studies that relate computation strategies with the type of problems proposed and other studies that characterize the strategies invented by students (Baek, 2006; Hartnett, 2007).

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Throughout this study we differentiate strategy from procedure – we consider that the pupils' procedures are the way in which they manipulate the numbers and that the strategies relate with the mathematical structure of that manipulation (Beishuizen, 1997).

To the present study it was particularly important to examine different categorisation of mental computation strategies and procedures. Here we highlight the categories proposed by Hartnett (2007), since they are comprehensive, being valid for all basic operations: count on and back, adjust and compensate, double and/or halve, break up numbers and use place value. Focusing on multiplication, Baek (2006) characterizes the following categories of invented strategies: direct modelling, repeated adding, doubling, partitioning number strategies and compensating strategies.

Several authors argue that pupils' understanding of multiplication evolves when faced with situations that emerge key aspects: the ideas, strategies and models (Fosnot & Dolk, 2001; Treffers & Buys, 2008). That evolution is also related with the way in which one promotes thinking and discussion during the tasks exploration. In the beginning of multiplication learning, pupils solve problems by counting groups, using repeated additions and then resorting to known multiplicative facts, as the concept of multiplication is being built (Treffers & Buys, 2008).

METHODOLOGY

The research focus on a teaching experiment (Gravemeijer & Cobb, 2006) carried out with a third grade class with 23 pupils, during eight months. The teaching experiment underlies the creation and exploration of a multiplication learning trajectory. It was conceived considering that learning mathematics with understanding (NCTM, 2007) includes moments of social interaction and moments of individual work, and implies the creation of a specific class culture with an investment in specific social and socio-mathematical norms (Cobb, Stephan, McClain & Gravemeijer, 2001).

Eleven sequences of multiplication tasks (division tasks were also included, favouring the relation between division and multiplication) were created and implemented, considering the learning objectives, the learning hypothesis and the established class culture (Simon, 1995). They were developed having as reference the multiplication learning landmarks (Fosnot & Dolk, 2001; Treffers & Buys, 2008). The learning experience was developed over 30 lessons of 2 hours each and the eleven task sequences of multiplication were implemented and videotaped. All written resolutions of the tasks were collected and analysed.

The learning trajectory was adapted, in a cyclic and continuous process, given its hypothetical nature and the unpredictable factors that arise in class (Simon, 1995). The tasks were adjusted by the researcher (the report's first author) and by the teacher, considering the procedures used by pupils while they worked on the previous tasks.

The inventory and description of the procedures used by the pupils in solving the tasks were carried out from the analysis of theirs written resolutions of the eleven sequences of multiplication tasks. The analysis of the procedures evolvement also considers class interactions. In order to analyse the pupils' procedures and their evolution, the tasks were grouped, according to their main characteristics, as presented in the below table.

Multiplication tasks (whole numbers)	T.1, T.2, T.4, T.7, T.8A, T.8B, T.10
Multiplication tasks (decimals)	T.13, T.16, T.17
Division tasks (whole numbers)	T.19, T.20 T.25, T.26
Multiplication tasks (involving a rate) (decimals)	T.29, T.30

Table 1: Grouping of tasks according to their characteristics

RESULTS

Procedures diversity

The pupils resort to assorted procedures when they are resolving multiplication tasks. These procedures were inventoried and grouped into global categories of procedures: counting, additive, subtractive, and multiplicative. In each of these categories we also identified and categorised specific procedures, organised in the below table.

Categories	Specific procedures	Frequency
Counting procedures	Skip-counting	13
Additive procedures	Using repeated addition	141
	Adding two terms	142
	Column calculation	14
Subtractive procedures	Using repeated subtraction	34
Multiplicative procedures	Using known facts	216
	Using doubles	128
	Using landmarks multiples	94
	Partitioning a number into nondecade numbers	190
	Partitioning a number into decade number	63
	Using compensating	31
	Doubling and halving	27
	Multiplying successively from a product of reference	30
	Column calculation	49

Table 2: Categories and procedures used by pupils

Evolution of the procedures

The analysis focused on the procedures used by pupils throughout the year has enabled us to highlight three aspects. The first aspect is related to the fact that each pupil uses several procedures to do the same computation. The second aspect concerns the frequency in the use of certain procedures, and the third aspect concerns the preference of some pupils for some procedures.

In the first stage of the teaching experiment, many pupils solve one task in more than one way. This seems to be strongly linked with their lack of security in the use of multiplicative procedures, frequently choosing to present a multiplicative procedure followed by its substantiation in additive terms. Also, as we illustrate by task 7 – Drapes, where one has to calculate the total number of flowers – the pupils seemed to enjoy demonstrating that they were able to perform each computation in many ways.

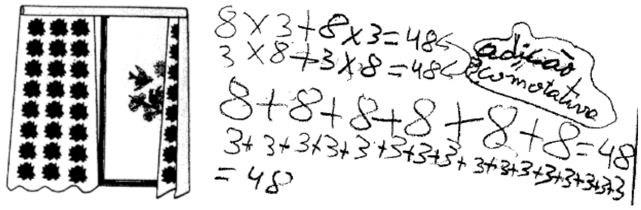

Figure 1: Cristóvão’s solution of subtask 2 – task 7

Cristóvão’s solution, for instance, includes two types of procedures – resorting to non decimal decompositions of one of the factors of the multiplication (8×6 or 6×8) and the use of addition, changing the part that is repeated. Apparently, the pupil has solved the problem by using the procedure associated to the first representation and the following ones are simply a way of demonstrating that the problem can also be solved in different ways.

Throughout the teaching experiment, the presentation of several procedures for the same problem becomes scarcer as the pupils realise that the point is solving the problem in the most adequate way.

A second aspect revealed by the analysis concerns the frequency of pupils’ use of certain procedures – some are often used, and others are rarely used. We have identified a high frequency in the use of: additive procedures, using multiplication known facts and partitioning a number. The use of relationships of double and half stand out by their low frequency, within the multiplicative procedures.

The additive procedures are predominant in the resolution of the first tasks, less used in the following tasks, and again more used in the final tasks. One of the reasons why they are highly frequent in the initial tasks seems to be related to the confidence the pupils feel by using them. Also the magnitude of the numbers involved (product less than 100) does not yet show their ineffectiveness, causing them to prevail for some time. Then, in solving tasks related to the array they emerge and seem to be consolidated in procedures related to the properties of the multiplication.

The solving of multiplication tasks in its proportional sense with numbers in the decimal representation shows a frequent return to the additive procedures. One of the reasons why this happened seems to be related with the organization of the data in tables, thereby suggesting numerical “compositions” and “decompositions” through addition. Also, computation started to include rational numbers represented in the decimal form, which are more difficult for pupils.

Frequently, and especially in the beginning of the year, the pupils also resort to known products. In effect, the products involved in the tasks are the mastering of the multiplication tables, already memorised, and so they do not need to perform any computation.

A part from the procedures which we have already identified, the pupils often resort to decimal or other decompositions of one of the factors. We present the example of task 10 – Stacks of boxes - in which several pupils use this type of procedure to determine the total number of apples in the 25 boxes with 48 apples each. Cristóvão and Hugo use a procedure of non decimal decomposition, suggested by the display of the boxes in the picture which is included in the task.

Figure 2: Cristóvão and Hugo's solution of subtask 2 – task 10

The pupils use three partial products which correspond to the decomposition of 25, according to the number of boxes in each two rows. Then they use the decimal decompositions which seem to them the easiest in the remaining calculus.

A part from the procedures that stand out as the most used, we also have the ones which are rarely used. Doubling and halving are among the least used procedures by the pupils, even though the numbers in the tasks seem to suggest them.

Finally, a third aspect is related with the preference of some pupils for specific procedures. Despite the multiple procedures presented and the evolution pupils shown, there are still some children that favour one in particular, even if it does not prove to be the best suited for a specific task.

Ana and Miguel have often chosen to multiply successively from a reference product, even after their peers had evolved to other procedures which were faster and more powerful. Figure 3 shows their solution of task 26 where one has to learn how to equally divide 256 animal creatures by 8 children.

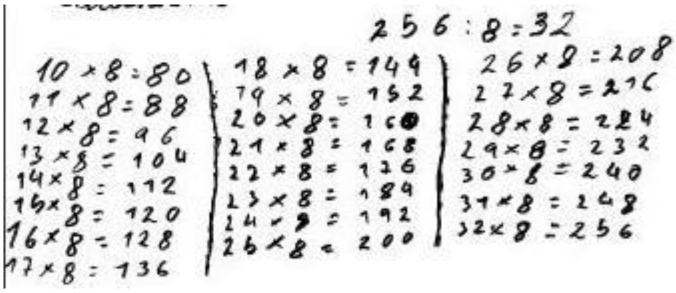

Figure 3: Ana and Miguel's solution of a part of task 26

The pupils start with the reference product 10×8=80 and systematically calculate all of the multiples of 8 until 32×8=256. Since this procedure has proven effective in previous tasks, they seem to feel confident about its use. In class, their procedure is discussed in the collective discussion.

> Ana – We did the whole multiplication table up until 32.
> Guilherme – But how did you know it was going to stop there? How many times did you do the multiplication table?
> Ana – We knew it because 32×8 is 256, the number of the miniatures.
> Teacher –Is it easy to do it this way?
> Ana – More or less. […] Because it takes some time and if we don't know the multiplication table, we can do a mistake.

Ana describes her reasoning and also justifies how she "knows when to stop", demonstrating that she understands the procedure she is using. She is also able to identify its risks.

Evolution and contexts

The previously discussed aspects, and globally the pupils' evolution cannot be separated from the contexts of the proposed tasks and the way in which whole class discussions have been organized. In this communication we are only focusing on the first aspect, emphasising that certain contexts and numbers that have been included have proven to be particularly suited for the evolution of the pupils' procedures. For example, the context of task 7, by appealing to the array and only showing half of the total to be calculated, was decisive in bringing forward multiplicative procedures, such as the ones that are associated with the decompositions of one of the factors and the use of doubles relationships. In turn, the context of task 10 has helped the consolidation of multiplicative procedures, especially with the ones that use the decomposition of one of the factors. The numbers included in the contexts have also influenced the pupils' evolution.

DISCUSSION

The results from this study show that the pupils resort to different procedures when solving multiplication tasks, gradually evolving from counting and additive procedures into multiplicative procedures, based on operation properties as referred by Fosnot and Dolk (2001), and Treffers and Buys (2008). This evolution is strongly anchored in the contexts of the tasks and in the numbers that had to be manipulated.

Pupils often use decimal or other decomposition procedures of one of the factors as it is referred in the investigation carried out by Baek (2006) – in which the pupils used mainly partition strategies associated with one or two factors of the multiplication. This author also found that the procedures which use double and half relations are the least used in the research that she conducted.

In the beginning, some pupils often use different procedures in order to perform a calculus. There are even some pupils who demonstrate a clear preference for the systematic use of a certain procedure, an aspect also identified by Gilmore and Papadatou-Pastou (2009).

This study has implications in the curricular development of multiplication. In fact, the analysis of the pupils' procedures and their evolution may guide the teaching sequence and assist teachers to decide "what comes next" by analysing the pupils' solutions (Ell, 2001). Also, the knowledge about the procedures used by pupils in the resolution of multiplication tasks and the way in which they evolve is a contribution towards defining effective learning trajectories.

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