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# Invertible Flexible Matrices

Júlia Justino[1,2,a)]

[1]*Department of Mathematics, Setubal School of Technology, Campus do IPS, Estefanilha, 2914-508 Setúbal, PORTUGAL*
[2]*Research Center for Mathematics and Applications (CIMA), Colégio Luís Verney, Rua Romão Ramalho, 59, 7000-671 Évora, PORTUGAL*

[a)]julia.justino@estsetubal.ips.pt

**Abstract.** Matrices with coefficients having uncertainties of type $o(.)$ or $O(.)$, called flexible matrices, are studied from the point of view of nonstandard analysis. The uncertainties of the afore-mentioned kind will be given in the form of the so-called neutrices, for instance the set of all infinitesimals. Since flexible matrices have uncertainties in their coefficients, it is not possible to define the identity matrix in an unique way and so the notion of spectral identity matrix arises. Not all nonsingular flexible matrices can be turned into a spectral identity matrix using Gauss-Jordan elimination method, implying that that not all nonsingular flexible matrices have the inverse matrix. Under certain conditions upon the size of the uncertainties appearing in a nonsingular flexible matrix, a general theorem concerning the boundaries of its minors is presented which guarantees the existence of the inverse matrix of a nonsingular flexible matrix.

## INTRODUCTION

In classical mathematics does not exist a very highly developed algebra of propagation of errors. In interval calculus [1] error operations are well defined but often computed with complex calculations or at a certain point the error bounds are so large that they no longer have practical value (see [2] and [3]).

This paper presents conditions that guarantee the existence of the inverse matrix of a flexible matrix, a matrix with coefficients having uncertainties of type $o(.)$ or $O(.)$; instead of using this functional form of neglecting an alternative formulation within nonstandard analysis, using sets of infinitesimals knowned as neutrices, will be used. Neutrices were introduced by the program of Van der Corput in [4].

Not all nonsingular flexible matrices can be turned into the identity matrix using Gauss-Jordan elimination operations. So not all nonsingular flexible matrices have the inverse matrix. Under certain conditions concerning the size of the uncertainties appearing in a flexible matrix it is possible to delimit all its minors and guarantee the existence of the inverse matrix.

The setting of this paper it is the axiomatic nonstandard analysis $IST$ as presented by Nelson in [5] and therefore the notions of *infinitesimal*, *infinitely large*, *limited* (real numbers which are not infinitely large) and *appreciable* (limited numbers which are not infinitesimals) numbers will be used.

## NEUTRICES AND EXTERNAL NUMBERS

External numbers were introduced in 1995 by Koudjeti and Van den Berg in Koudjeti's thesis and a chapter of "Nonstandard Analysis in Practice" [6] (Springer, F. and M. Diener, eds.), to serve as mathematical models of orders of magnitude within nonstandard analysis.

Within $IST$ the nonstandard numbers are already present in the standard set $\mathbb{R}$.

**Definition 1** *A* neutrix *is an additive convex subgroup of* $\mathbb{R}$, *which is symmetric with respect to* 0.

**Example 1** *Except for* $\{0\}$ *and* $\mathbb{R}$ *itself, all neutrices are external sets (with internal elements). The most common neutrices are £, the external set of all limited numbers and* $\oslash$*, the external set of all infinitesimal numbers.*

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So neutrices can be seen as a sort of generalized zeros that have the following properties:

- Neutrices are totally ordered by inclusion:

$$\{0\} \subset \varepsilon^3 £ \subset \varepsilon^2 \oslash \subset \varepsilon^2 £ \subset \varepsilon\oslash \subset \varepsilon £ \subset \oslash \subset £ \subset \omega\oslash \subset \omega £ \subset \mathbb{R},$$

  with $\varepsilon$ an infinitesimal and $\omega$ an infinitely large number.
- Neutrices are invariant under multiplication by appreciable numbers.
- The sum of two neutrices is the largest one for inclusion.

**Definition 2** *An* external number $\alpha$ *is the algebraic sum of a real number $a$ with a neutrix $A$* :

$$\alpha = a + A.$$

- $A$ is called the *neutrix part of* $\alpha$ and is unique but $a$ is not, because for all real number $c \in \alpha$, $\alpha = c + A$. So $a$ is called a *representative of* $\alpha$.
- $\alpha$ is called *zeroless* if it is not a neutrix, so $0 \notin \alpha$.

**Example 2** *Let $\varepsilon$ be a positive infinitesimal. Then $1 + \varepsilon £$, $\frac{1}{\varepsilon} + \oslash$, $\varepsilon$ and £ are external numbers. Only £ is not zeroless. Furthermore, $\varepsilon £$ is the neutrix part of* $1 + \varepsilon £$ *and* 1 *or* $1 - \varepsilon$ *are representatives of* $1 + \varepsilon £$. *All classical real numbers are external numbers with the neutrix part equal to* $\{0\}$.

Let $\alpha = a + A$ and $\beta = b + B$ be two external numbers. Then $\alpha$ and $\beta$ are either disjoint or one contains the other. In addition,

1. $\alpha \pm \beta = a \pm b + \max(A, B)$ ;
2. $\alpha \cdot \beta = ab + \max(aB, bA, AB) = ab + \max(aB, bA)$ , if $\alpha$ or $\beta$ is zeroless;
3. $\frac{\alpha}{\beta} = \frac{a}{b} + \frac{1}{b^2}\max(aB, bA) = \frac{\alpha \cdot \beta}{b^2}$, with $\beta$ zeroless.

The *absolute value of* $\alpha$ is defined by

$$|\alpha| = \begin{cases} \alpha & , \quad A \leq \alpha \\ -\alpha & , \quad \alpha < A \end{cases}.$$

The next tables present the principal rules of external calculus used in this paper:

| ± | ⊘ | @ | £ |
|---|---|---|---|
| ⊘ | ⊘ | @ | £ |
| @ | @ | @ | £ |
| £ | £ | £ | £ |

| × | ⊘ | @ | £ |
|---|---|---|---|
| ⊘ | ⊘ | ⊘ | ⊘ |
| @ | ⊘ | @ | £ |
| £ | ⊘ | £ | £ |

Being stable for some translations, additions and multiplications, external numbers are models of orders of magnitude with imprecise boundaries.

## NONSINGULAR FLEXIBLE MATRICES

**Definition 3** *Let $n \in \mathbb{N}$ be standard and let $\alpha_{ij} = a_{ij} + A_{ij}$ for all $i, j \in \{1, ..., n\}$. We call*

$$\mathcal{A} = \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \dots & \alpha_{nn} \end{bmatrix} = \left[a_{ij}\right]_{n\times n} + \left[A_{ij}\right]_{n\times n}$$

*a* flexible matrix *if $A_{ij} \neq \{0\}$ for some $i, j \in \{1, ..., n\}$. If* $\det \mathcal{A}$ *is zeroless, $\mathcal{A}$ is called* nonsingular.

**Example 3**

$$\mathcal{A} = \begin{bmatrix} 3+\varepsilon\oslash & -1+\oslash \\ 2+\varepsilon£ & 1+\varepsilon\oslash \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} \varepsilon\oslash & \oslash \\ \varepsilon£ & \varepsilon\oslash \end{bmatrix}$$

*is a nonsingular flexible matrix because* $\det\mathcal{A} = 5 + \oslash$ *is zeroless.*

Since flexible matrices have uncertainties in their coefficients, it is not possible to define the identity matrix in an unique way.

**Definition 4** *Let $n \in \mathbb{N}$ be standard. Any $n \times n$ matrix infinitesimal close to the identity matrix is called a* spectral identity matrix*:*

$$\begin{bmatrix} 1+A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & 1+A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & 1+A_{nn} \end{bmatrix}_{n\times n},$$

*with $A_{ij} \subseteq \oslash$ for all $i, j \in \{1, ..., n\}$.*

**Example 4** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1+\varepsilon\oslash & \oslash \\ \varepsilon£ & 1+\oslash \end{bmatrix}$ *are spectral identity matrices but* $\begin{bmatrix} 1+\varepsilon\oslash & \frac{\oslash}{\varepsilon} \\ \varepsilon & 1+\oslash \end{bmatrix}$ *is not since $\varepsilon$ is not a neutrix and $\frac{\oslash}{\varepsilon} \nsubseteq \oslash$.*

**Definition 5** *Let $\mathcal{A}$ be a nonsingular flexible matrix. If it is possible to turn $\mathcal{A}$ into a spectral identity matrix using Gauss-Jordan elimination operations, in particular swapping rows, multiplying rows by zeroless limited numbers and adding a multiple of one row to another row, $\mathcal{A}$ is called an* invertible flexible matrix.

**Example 5** *The flexible matrix* $\mathcal{A} = \begin{bmatrix} 3+\varepsilon\oslash & -1+\oslash \\ 2+\varepsilon£ & 1+\varepsilon\oslash \end{bmatrix}$ *is invertible. In fact,*

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} 3+\varepsilon\oslash & -1+\oslash \\ 2+\varepsilon£ & 1+\varepsilon\oslash \end{bmatrix} \xrightarrow{\frac{1}{3}L_1} \begin{bmatrix} 1+\varepsilon\oslash & -\frac{1}{3}+\oslash \\ 2+\varepsilon£ & 1+\varepsilon\oslash \end{bmatrix} \\ &\xrightarrow[L_2-2L_1]{} \begin{bmatrix} 1+\varepsilon\oslash & -\frac{1}{3}+\oslash \\ \varepsilon£ & \frac{5}{3}+\oslash \end{bmatrix} \xrightarrow[\frac{3}{5}L_2]{} \\ &\begin{bmatrix} 1+\varepsilon\oslash & -\frac{1}{3}+\oslash \\ \varepsilon£ & 1+\oslash \end{bmatrix} \xrightarrow{L_1+\frac{1}{3}L_2} \begin{bmatrix} 1+\varepsilon£ & \oslash \\ \varepsilon£ & 1+\oslash \end{bmatrix}. \end{aligned}$$

**Example 6** $\mathcal{B} = \begin{bmatrix} 2 & -1+\varepsilon£ \\ 2\varepsilon & \varepsilon+\varepsilon\oslash \end{bmatrix}$ *is a nonsingular flexible matrix because* $\det\mathcal{B} = 4\varepsilon + \varepsilon\oslash$ *is zeroless. However, $\mathcal{B}$ is not an invertible matrix. In fact,*

$$\begin{aligned} \mathcal{B} &= \begin{bmatrix} 2 & -1+\varepsilon£ \\ 2\varepsilon & \varepsilon+\varepsilon\oslash \end{bmatrix} \xrightarrow[L_2-\varepsilon L_1]{} \begin{bmatrix} 2 & -1+\varepsilon£ \\ 0 & 2\varepsilon+\varepsilon\oslash \end{bmatrix} \\ &\xrightarrow[\frac{1}{2}L_1]{} \begin{bmatrix} 1 & -\frac{1}{2}+\varepsilon£ \\ 0 & 2\varepsilon+\varepsilon\oslash \end{bmatrix} \end{aligned}$$

*and it is not possible to turn this last matrix into a spectral identity matrix using Gauss-Jordan elimination operations since $\frac{1}{2\varepsilon}$ is not a limited number.*

So not all nonsingular flexible matrices are invertible. The condition that guarantees the existence of the inverse matrix, introduced by Justino and Van den Berg in [7], is linked to the control of the uncertainties appearing in the coefficients of a flexible matrix so that the error propagation of its minors will be limited. In fact, during the condensation of matrix $\mathcal{A}$, its coefficients are progressively turned into minors of $\mathcal{A}$. Besides, when dividing a row by its pivot, the value of the pivot cannot be too small, otherwise the uncertainties of the row may expand in a way that $\mathcal{A}$ cannot be turned into a spectral identity matrix.

**Definition 6** *Let $\mathcal{A} = \left[\alpha_{ij}\right]_{n\times n}$ be a nonsingular flexible matrix, with $\alpha_{ij} = a_{ij} + A_{ij}$ for all $i, j \in \{1, ..., n\}$. The* relative uncertainty of $\mathcal{A}$ *is defined by the neutrix*

$$R(\mathcal{A}) = \frac{\overline{A}\cdot\overline{\alpha}^{n-1}}{\Delta},$$

*where $\overline{A}$ is the maximum of the uncertainties of $\mathcal{A}$, $\overline{\alpha}$ is the maximum coefficient on matrix $\mathcal{A}$ and $\Delta = \det \mathcal{A}$.*

**Example 7** *Let $\mathcal{A} = \begin{bmatrix} 3+\varepsilon\oslash & -1+\oslash \\ 2+\varepsilon£ & 1+\varepsilon\oslash \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} 2 & -1+\varepsilon£ \\ 2\varepsilon & \varepsilon+\varepsilon\oslash \end{bmatrix}$. So*

$$R(\mathcal{A}) = \frac{\overline{A}\cdot\overline{\alpha}}{\Delta} = \frac{\oslash\cdot(3+\varepsilon\oslash)}{5+\oslash} = \oslash$$

*and*

$$R(\mathcal{B}) = \frac{\overline{B}\cdot\overline{\beta}}{\Delta} = \frac{\varepsilon£\cdot 2}{4\varepsilon+\varepsilon\oslash} = \frac{\varepsilon£\cdot(4\varepsilon+\varepsilon\oslash)}{16\varepsilon^2} = \frac{\varepsilon^2£}{16\varepsilon^2} = £.$$

**Notation 1** *Let $\mathcal{A} = \left[\alpha_{ij}\right]_{n\times n}$ be a nonsingular flexible matrix, with $\alpha_{ij} = a_{ij} + A_{ij}$ for all $i, j \in \{1, ..., n\}$, and let $k \in \{1, \ldots, n-1\}$. We denote:*

1. *$[\mathcal{A}]_{i_1\cdots i_k, j_1\cdots j_k}$ as the $(n-k)\times(n-k)$ matrix formed by removing from $\mathcal{A}$ the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and $1 \leqslant j_1 < \cdots < j_k \leqslant n$;*
2. *$\left|\overline{M_k}\right| = \max\limits_{\substack{2\leqslant i_1<\cdots<i_k\leqslant n \\ 2\leqslant j_1<\cdots<j_k\leqslant n}} \left|\det[\mathcal{A}]_{i_1\cdots i_k, j_1\cdots j_k}\right|$ as the maximum $k^{th}$ minor of $\mathcal{A}$;*
3. *$\overline{m_k}$ as a representative of $\overline{M_k}$.*

The next theorem states that condition $R(\mathcal{A}) \subseteq \oslash$ guarantees the delimitation of all minors of $\mathcal{A}$, which are used to calculate the determinant of $\mathcal{A}$ and the inverse matrix.

**Theorem 1** *Let $\mathcal{A} = \left[\alpha_{ij}\right]_{n\times n}$, with $\alpha_{ij} = a_{ij} + A_{ij}$ for all $i, j \in \{1, ..., n\}$, such that $R(\mathcal{A}) \subseteq \oslash$. Then, for all $k \in \{1, \ldots, n-1\}$,*

$$\oslash\Delta < |\overline{m_k}| \leqslant £.$$

**Corollary 1** *Let $\mathcal{A}$ be a nonsingular flexible matrix. If $R(\mathcal{A}) \subseteq \oslash$, then $\mathcal{A}$ is invertible.*

**Example 8**
1. *$\mathcal{A} = \begin{bmatrix} 3+\varepsilon\oslash & -1+\oslash \\ 2+\varepsilon£ & 1+\varepsilon\oslash \end{bmatrix}$ is invertible and $R(\mathcal{A}) = \oslash$.*
2. *$\mathcal{B} = \begin{bmatrix} 2 & -1+\varepsilon£ \\ 2\varepsilon & \varepsilon+\varepsilon\oslash \end{bmatrix}$ is not invertible and $R(\mathcal{B}) = £ \nsubseteq \oslash$.*

This final result implies that, although not all nonsingular flexible matrices are invertible, if the relative uncertainty of a nonsingular flexible matrix is infinitesimal, the error propagation of its minors will be limited, ensuring that it can be turned into a spectral identity matrix by Gauss-Jordan elimination which guarantees the existence of the inverse matrix.

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