# CONSTRUCTING MULTIPLICATION: DIFFERENT STRATEGIES USED BY PUPILS

**Joana Brocardo** **Lurdes Serrazina** **Isabel Rocha**

*ESE de Setúbal* *ESE Lisboa* *ESE Leiria*

**Abstract**

*The Project "Number sense development: curricular demands and perspectives" aims to study the development of number sense in elementary school (5 to 12 years old). This paper presents a discussion based on one of the six case studies developed by the project. We will focuses on the strategies used by 7-years old pupils when solving multiplication problems, namely on the awareness of existence of different strategies and the inclination to utilize an efficient representation or method.*

## Introduction

Number sense has been considered one of the most important components of elementary mathematics curriculum. The development of personal strategies of calculation and its implications to solve problems in real situations are recommended by both international literature (Fuson, 2003) and Portuguese curricular documents.

The term number sense has been used by several researchers to mean a group of numerical competencies that, nowadays, are considered very important to develop with students. For us, the meaning of this term, adopted by McIntsoh, Reys and Reys (1992) includes all the main points. They consider that number sense comprehends:

1. Knowledge and facility with numbers, witch includes multiple representations of numbers, recognizing the relative and absolute magnitudes of numbers, composing and decomposing numbers and selecting and using benchmarks.
2. Knowledge and facility with operations, witch includes the understanding of the effects of operations on numbers, the understanding and the use of the operations properties and their relationships.
3. Applying knowledge of and facility with numbers and operations to computational settings, witch includes the understanding to make connections between the context of a situation and the computation procedures, requiring knowledge of multiple computational strategies.

Most countries have emphasized during the last twenty years, the development of number sense together with the development of strategies and computation procedures and their flexible applications both in real practice contexts and in new ones. In NCTM Standards, understanding number and operations, developing number sense and gaining fluency in arithmetic computation form the core of mathematics education for the elementary grades (NCTM, 2000). The Project *National Numeracy Strategy* also points out these ideas when states that England changed the way in which mathematics is taught in many schools, with a new emphasis on mental mathematics, with new teaching approaches to help children develop a repertoire of computational skills involving work on mental calculations and strategies. (Askew and Ebbutt, 2000).

Instead, Portuguese elementary school tradition emphasizes arithmetic algorithms and that way of facing number sense is far away from it. Some studies reflected on the effect of teaching written algorithms on the development of children's mental strategies and number sense (Clarke, 2004). Encourage students to use only one method to solve problems limits their capacity to use flexible and creative thinking. In counterpoint, Clarke shows the benefits of developing concepts and strategies for mental computation prior to more formal written computation.

A closely relation between research and practice is seen as an important element to help changing teacher's practice. It is very important to work with teachers, to develop new materials with them, to refine these materials according to instruction experiments results in a process that leads to construct a conjectured local instruction theory (Gravemeijer, 1997).

## The project *Number sense development: curricular demands and perspectives*

The project *Number sense development: curricular demands and perspectives* aims to study the development of number sense in elementary school (5-12 years old). The team Project is composed by classroom teachers and researchers who work collaboratively in all phases of research. We developed related sequence of 3 or 4 tasks – *task chain* - and implemented them in a particular classroom, covering different grades from kindergarten (5 years old ) to 5th grade (11 years). Each task chain was developed as an hypothetical learning trajectory in the sense used by Simon (1995).

Therefore, one of the components of the project *Number sense development: curricular demands and perspectives,* is to develop chains of learning activities that supports the development of number sense. Those chains has been thinking with the support of Simon`s model. For Simon, the teacher in order to plan his teaching, has to make decisions about contents and about the learning tasks. So, in this context, it is introduced by this author, the concept of *hypothetical learning trajectory*, a cycle of learning assigned with some tasks constructed by the teacher, attending to the mathematical ideas and processes that he intends to develop in the students.

The learning trajectory is hypothetical because is thought as experimental and because is not possible to be sure that it will be a real and efficient way of learning. Of course, it is possible to make previsions because the teacher may anticipate the approaches, the discussions and solutions that may be stimulated by the potential of the tasks.

It was in this context, that the task chain experimented in the case study presented in this paper, was developed and implemented.

In addition the project has three other objectives:

- To understand the main difficulties faced by pupils when they develop number sense;
- To study the curricular integration of these activities and the demands they pose to teachers;
- To identify professional practices that facilitates number sense development.

We developed six qualitative case studies. Each case study analyzed the implementation in a particular classroom of a related sequence of 3 or 4 tasks – *task chain* - and covered different grades from kindergarten (5 years) to 5th grade (11 years).

## Methodology

We will present a discussion based on the analysis of one of the case studies developed by the project and implemented in a second grade classroom (20 seven years old children). These pupils had already worked addition and subtraction with numbers up to 100.

The focus of the *task chain* experimented in this case, is a learning trajectory to develop the multiplication concept. It is composed by four tasks that were presented in the last semester of school year. This work was developed along four weeks, one task per week, explored every Monday morning.

Each Monday morning the classroom teacher organized a brief oral introduction to each task. He mainly tried to capture pupils' attention to the context presented in each task. He also proposed a given period of time to explore the task in small groups. After that he organized a whole group discussion where all groups had the opportunity to share different approaches to the problems posed.

This *task chain* focus the construction of a learning trajectory, which starts from the knowledge related to additive computation to develop the multiplication concept. More precisely it deals with relation between some table products and the understanding of specific properties of multiplication. This learning trajectory was also foreseen to introduce the double number line model and to enhance the concept of multiplication relating it with the rectangular model.

Classes where tasks were developed were videotaped by one of the members of the research team, who also took some written field notes from her class observations. After that all videotapes were watched and more meaningful episodes transcribed and completed with field notes.

Data analysis was made task by task, first a description how task was presented to pupils, following how it was explored and discussed in classroom. It also includes the final discussion when it was not embedded with the exploration. It ends with a synthesis of each task centred in the processes used by pupils according to categories about number sense, adapted from Mcintosh, Reys and Reys (1992).

After the analysis of each task of the chain, the hypothetical learning trajectory and the learning goals for each pupil, were confronted with the process followed by each one in order to try to establish a learning trajectory followed by these classroom pupils.

In this paper we focuses our analysis in the category - *Applying knowledge of and facility with numbers and operations to computational settings* - which includes: (1) understanding the relationship between problem context and the necessary computation, (2) awareness that multiple strategies exist and (3) inclination to utilize an efficient representation and/or method.

**Understanding the relationship between problem context and the necessary computation**

One of the conclusions of a previous study carried out by Brocardo, Serrazina, Kraemer (2003) shows a strong tendency of Portuguese pupils to a mechanical use of algorithms: they read the question and they ask themselves which computation should they do. They do not analyse the meaning of question and the answer they propose.

In this case was not observed this kind of procedure. Pupils tended to analyse the context and to relate it with the computation they performed. This tendency seems to be related with the teacher's concern to enhance pupils' interpretation of the proposed problem, as he did when he proposed the first task:

> Teacher: And now you can read what is presented in the task and, first of all we are going to interpret it, ok? Is there any word you can not understand?

At the beginning , although pupils had a concern to analyse the context, many times that analysis tended to be more important than the mathematical work related with the problem that was been discussed. For instance, in the first task, the discussion of the meaning of the word aperitif drove pupils to talk about meals (main course, desert, aperitif) and about the sort of aperitifs that they had already tasted.

In the following tasks this aspect was less relevant. The pupils did not take so much time to talk about task lateral aspects. However, they always made some comments to the task context. We will show in the next section the strategies they used.

**Awareness that multiple strategies exist**

Our data show a progressive awareness of different strategies. Pupils strategies repertoire was enriched along the work developed with this task chain.

In the first task pupils had to calculate the number of sausages' slices. Most of them used additive strategies:

1 sausage - 2 slices

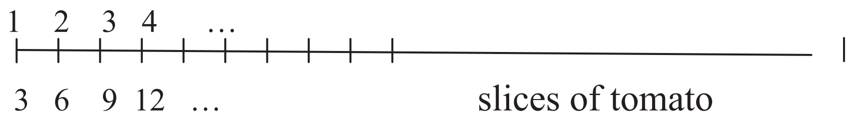
2 sausages - 4 slices

3 sausages - 6 slices

The additive strategy was also used in a more sophisticated way: two groups related the questions like this:

> If I need 24 slices of cheese then I have to add 24 to know the number of sausages.
>
> To know the number of slices of tomatoes I have to add 24 to the number of sausages.

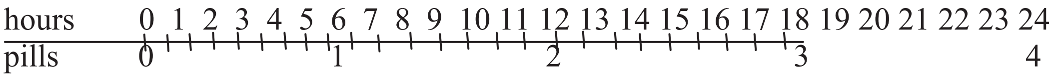
Some pupils had difficulties in organizing their thought. So, the teacher decided to introduce the double number line:



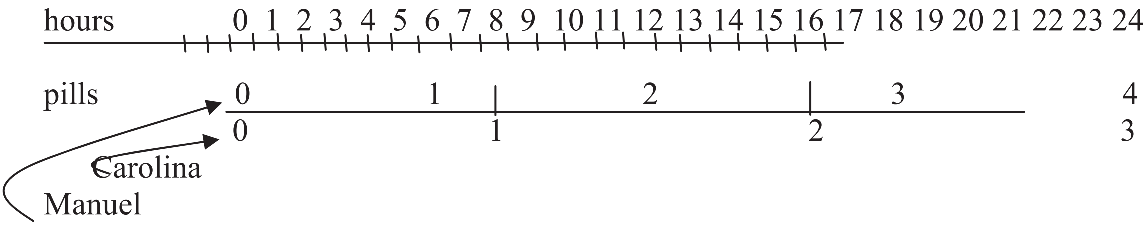

This strategy was easily understood by the pupils that begun to use it to solve some of the questions posed in the next tasks.

In the second task most of the pupils used this strategy, for instance, to find the number of boxes they need to pack 100 eggs:


This strategy was also used to support the explanations that teacher had to provide to some of the pupils. For instance, in the 3rd task, as some of the pupils had difficulties to express how they could explain that Manuel took 4 pills in one day, the teacher used the representation:



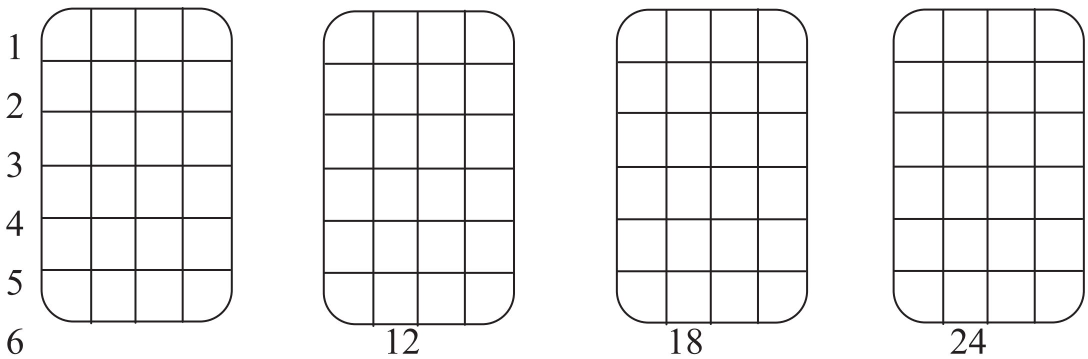

The double number line was also the support that teacher used to compare the number of pills that Carolina and Manuel took in one day.



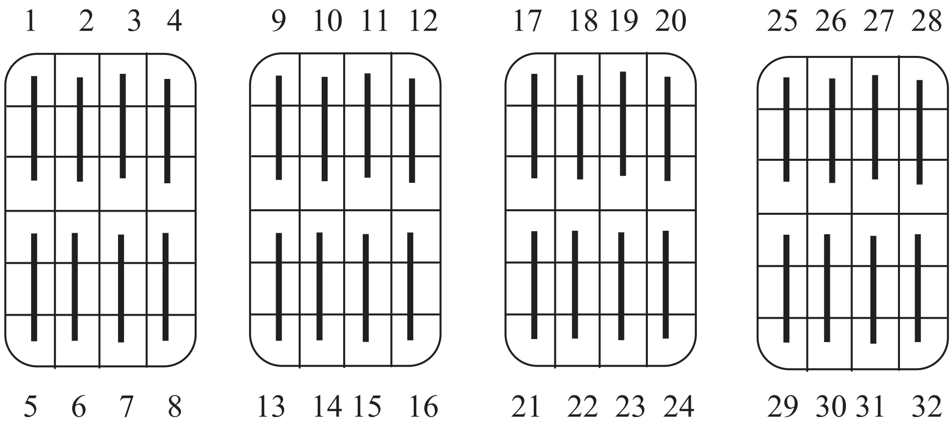

The concept of multiplication and its relation with the rectangular model was progressively developed. In the last task, when they were asked to propose different possibilities to pack 30 Chiclets, most pupils used the concept of multiplication relating it with the rectangular model and begun to present several solutions:

- 1x30
- 2x 15
- 3x 10
- 5x 6

The use of certain strategies seemed to be related with the context. For instance, the drawing of the blisters supported the strategies used by Francisco and his peer:



> Francisco: I already knew that Manuel took 4 pills per day and Carolina took 3 and then I did 4 pill boxes …. 4 blisters and then I did.
>
> Teacher: In one row.
>
> Francisco: In one row… aah… one day, and the other row it was the other day … (…)
>
> Francisco: Because the rows have 4 pills and it was 24, he needed 24 days to take all the pills.
>
> (…)
>
> Francisco: And then I did the same thing …but Carolina took 3 pills per day and I did this:




**Inclination to utilize an efficient representation and/or method**

Besides the progressive awareness of different strategies, an aspect which was emphasized by teacher was the use of more efficient strategies.

Data shows that this is a more difficult level for most of the pupils. Many of them can persist to use a long process – like jumping one by one in the double number line.

During the two first tasks only one group seemed to be interested in thinking in the "most efficient process". However the reflection proposed by the teacher when whole class analyses the different strategies used by the small groups, seemed to be an important way of facilitating the inclination to use a more sophisticated strategy.

An example of this was the discussion that Tiago and Rodrigo strategy facilitated:

> To calculate the number of days that Manuel and Carolina needed to take all pills they wrote on the blackboard this:

| Manuel | Carolina |
|---|---|
| 1 day 24 h | 1 day 24 h |
| 6 days one blister | 8 days one blister |

12 days two blisters 16 days two blisters

24 days four blisters 32 days four blisters

The teacher helping whole class to understand what Tiago and Rodrigo presented commented:

> Not to jump one by one …you may do large jumps …for 6 days how many pills? … 24 …24 with 24 is … right! Can you see? Now you have to do another jump. 6 days more is ? … So, more 24 … do bigger jumps.

and used this representation.

| | | | | |
|---|---|---|---|---|
| Days | 0 | 6 | 12 | 24 |
| Blist. | 0 | 1 | 2 | 4 |

**Concluding remarks**

We tried to illustrate that along the exploration of these 4 tasks, pupils develop a clear and powerful understanding of multiplication. They begun, in first task, to use additive strategies that corresponded to an activity in the task setting: interpretation and solutions depend on understanding of how to act in the setting (Gravemeijer, 2005). During discussion the teacher facilitated the change of ideas and strategies among pupils. He also had the initiative to introduce different approaches that facilitated the development of more powerful strategies.

In the second task most of the pupils already used models of the explored situation as an approach to the task.

In the fourth task pupils clearly used different models and they could relate multiplication with the rectangular model.

According to Dolk and Fosnot (2001), counting one by one is not multiply. The development of new strategies as doubling and the use and understanding of properties of multiplication facilitate the growing capacity of children thinking in terms of number relations and enhance the number sense. In this sense we can say that these pupils developed number sense: they could use different models; they could begin to use the ones that were more "powerful"; and they could relate multiplication with rectangular model. However not all students were able to reach the same flexibility and level of understanding. For some of them successive addition and jumping by ones continued to be strategies they preferred (or were able) to use. The context seems to have an important role in this process.

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