Cramer’s Rule applied to Flexible Systems of Linear Equations

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1 Introduction

In this study we will find the conditions that guarantee the existence of a maximal solution on systems of linear equations which have coefficients having uncertainties of type $o(\cdot)$ or $O(\cdot)$. We will not use the functional form of neglecting knowned as $o(\cdot)$ or $O(\cdot)$ but an alternative formulation within nonstandard analysis using sets of infinitesimals knowned as neutrices, introduced by the program of Van der Corput in [1]. This kind of systems will be called flexible systems of linear equations.

Summary:

1. Brief recall of external numbers, in which we distinguish neutrices, a sort of generalized zeros.

2. Defining what is a flexible system of linear equations. Some examples of the application of Cramer’s Rule.

3. Presentation of a general theorem that guarantees the existence of a maximal solution.
2 External numbers

External numbers are were introduced in 1995 by Koudjeti and Van den Berg in Koudjeti’s thesis [2] and a chapter of "Nonstandard Analysis in Practice" [3] (Springer, F. and M. Diener, eds.), to serve as mathematical models of orders of magnitude within nonstandard analysis.

We will use the notions of infinitesimal, limited and appreciable numbers. *Infinitesimal numbers* (or *infinitesimals*) are infinitely small real numbers, nearly equal to zero, *limited numbers* are real numbers which are not infinitely large and *appreciable numbers* are limited numbers which are not infinitesimals.
2.1 Neutrices

A *neutrix* is an additive convex subgroup of $\mathbb{R}$, which is symmetric with respect to 0.

Except for $\{0\}$ and $\mathbb{R}$ itself all neutrices are external sets (with internal elements). The most common neutrices are $\mathcal{L}$, the external set of all limited numbers and $\emptyset$, the external set of all infinitesimal numbers.

Neutrices are totally ordered by inclusion:

\[
\{0\} \subset \ldots \subset \varepsilon^2 \emptyset \subset \varepsilon \emptyset \subset \varepsilon \mathcal{L} \subset \emptyset \subset \emptyset \mathcal{L} \subset \ldots \subset \mathbb{R},
\]

with $\varepsilon$ an infinitesimal number,

and are invariant under multiplication by appreciable numbers.

The sum of two neutrices is the largest one.
2.2 External numbers

An external number $\alpha$ is the algebraic sum of a real number $a$ with a neutrix $A$:

$$\alpha = a + A.$$

For a neutrix $A$, an external number $\alpha$ is called an absorver of $A$ if $\alpha A \subseteq A$.

**Example:** If $\epsilon$ is a positive infinitesimal, then $\epsilon$ is an absorver of $\emptyset$ because $\epsilon \emptyset \subseteq \emptyset$.

An external number $\alpha$ which is not a neutrix is called zeroless, and is denoted by $0 \notin \alpha$. Then

$$\frac{A}{a} \subseteq \emptyset.$$

Being stable for some translations, additions and multiplications, external numbers are models of orders of magnitude with imprecise boundaries.
3 Flexible system

Definition 1 Let standard $n \in \mathbb{N}$ and $\alpha_{ij} = a_{ij} + A_{ij}$, $\beta_j = b_j + B_j$, $\xi_j = x_j + X_j$ for all $i, j \in \{1, ..., n\}$. Consider the following system:

\[
\begin{array}{ccc}
\alpha_{11}\xi_1 + \ldots + \alpha_{1j}\xi_j + \ldots + \alpha_{1n}\xi_n = \beta_1 \\
\vdots \\
\alpha_{n1}\xi_1 + \ldots + \alpha_{nj}\xi_j + \ldots + \alpha_{nn}\xi_n = \beta_n
\end{array}
\]

If $A_{ij} \subseteq \varnothing$ and $B_j \subseteq \varnothing$ for all $i, j \in \{1, ..., n\}$, this system is called a flexible system of linear equations.

Example 2 Consider the following flexible system of linear equations, where $\epsilon$ is a positive infinitesimal:

\[
\begin{array}{ll}
(3 + \epsilon \varnothing) x + (-1 + \varnothing) y = 1 + \epsilon \mathcal{L} \\
(2 + \epsilon \mathcal{L}) x + (1 + \epsilon \varnothing) y = \epsilon \mathcal{L}
\end{array}
\]  \hspace{1cm} (1)

Notice that this system is very close (infinitesimal close) with the classical system \[
\begin{array}{ll}
3x - y = 1 \\
2x + y = 0
\end{array}
\] which has the
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exact solution \[
\begin{align*}
x &= \frac{1}{5} \\
y &= -\frac{2}{5}
\end{align*}
\] 
produced by Cramer’s Rule.

System (1) has the matricial representation given by 
\[A\chi = B, \text{ with } \chi = \begin{bmatrix} x \\ y \end{bmatrix} \] 
and

\[A|B = \begin{bmatrix} 3 + \epsilon \varnothing & -1 + \varnothing & 1 + \epsilon \mathcal{L} \\ 2 + \epsilon \mathcal{L} & 1 + \epsilon \varnothing & \epsilon \mathcal{L} \end{bmatrix}.
\]

\[\Delta = \det A = \begin{vmatrix} 3 + \epsilon \varnothing & -1 + \varnothing \\ 2 + \epsilon \mathcal{L} & 1 + \epsilon \varnothing \end{vmatrix} = 5 + \varnothing \neq 0
\]

The solution produced by Cramer’s Rule is:

\[x = \frac{\begin{vmatrix} 1 + \epsilon \mathcal{L} & -1 + \varnothing \\ \epsilon \mathcal{L} & 1 + \epsilon \varnothing \end{vmatrix}}{\Delta} = \frac{1+\epsilon \mathcal{L}}{5+\varnothing} = \frac{1}{5} + \varnothing \text{ and}
\]

\[y = \frac{\begin{vmatrix} 3 + \epsilon \varnothing & 1 + \epsilon \mathcal{L} \\ 2 + \epsilon \mathcal{L} & \epsilon \mathcal{L} \end{vmatrix}}{\Delta} = \frac{-2+\epsilon \mathcal{L}}{5+\varnothing} = -\frac{2}{5} + \varnothing.
\]

But when we verify the fidelity of this solution, we see that it is not valid because \(\epsilon \mathcal{L} \subset \varnothing\):
What is the problem then?

If we look at the uncertainties of the system, we notice that the maximum of the uncertainties on matrix $\mathcal{A}$ is $\varnothing$ while the minimum of the uncertainties on matrix $\mathcal{B}$ is $\epsilon \mathcal{L}$ and $\varnothing \nsubseteq \epsilon \mathcal{L}$.

Let $\overline{\mathcal{A}}$ be the maximum of the uncertainties appearing in the matrix of coefficients and $\underline{\mathcal{B}}$ be the minimum of the uncertainties appearing in the second member of a system $\mathcal{A} \mathcal{X} = \mathcal{B}$. We will see that it is essencial to assure that $\overline{\mathcal{A}} \subseteq \underline{\mathcal{B}}$ in order to guarantee the validity of the solution produced by Cramer’s Rule, which means that matrix $\mathcal{A}$ has to be more precise than matrix $\mathcal{B}$.
Example 3 Consider the following flexible system of linear equations, where $\epsilon$ is a positive infinitesimal:

\[
\begin{align*}
3x + (-1 + \epsilon \mathcal{O})y &= 1 + \epsilon \mathcal{L} \\
2x + y &= \epsilon \mathcal{L}
\end{align*}
\]

whose matricial representation is given by $A \mathbf{x} = B$, with $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A|B = \begin{bmatrix} 3 & -1 + \epsilon \mathcal{O} \\ 2 & 1 \end{bmatrix}$. \\

We have $\overline{A} = \epsilon \mathcal{O} \subset \epsilon \mathcal{L} = \overline{B}$ and

\[
\Delta = \det A = \begin{vmatrix} 3 & -1 + \epsilon \mathcal{O} \\ 2 & 1 \end{vmatrix} = 5 + \epsilon \mathcal{O} \neq 0.
\]

$\Delta$ is not an absorver of $\overline{B}$ ($\Delta \overline{B} = \epsilon \mathcal{L} = \overline{B}$) and $\overline{B} = \epsilon \mathcal{L} = \overline{B}$.

The solution produced by Cramer's Rule is:

\[
x = \frac{\begin{vmatrix} 1 + \epsilon \mathcal{L} & -1 + \epsilon \mathcal{O} \\ \epsilon \mathcal{L} & 1 \end{vmatrix}}{\Delta} = \frac{1 + \epsilon \mathcal{L}}{5 + \epsilon \mathcal{O}} = \frac{1}{5} + \epsilon \mathcal{L} \quad \text{and}
\]
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\[ y = \begin{vmatrix} 3 & 1 + \epsilon \ell \\ 2 & \epsilon \ell \end{vmatrix} \Delta = -2 + \epsilon \ell \quad \frac{5 + \epsilon \O}{5} = -\frac{2}{5} + \epsilon \ell, \]

and it is valid one as we can easily verify:

\[ 3x + (-1 + \epsilon \O) y = 3 \left( \frac{1}{5} + \epsilon \ell \right) + (-1 + \epsilon \O) \left( -\frac{2}{5} + \epsilon \ell \right) = 1 + \epsilon \ell \quad \text{and} \]

\[ 2x + y = 2 \left( \frac{1}{5} + \epsilon \ell \right) + \left( -\frac{2}{5} + \epsilon \ell \right) = \epsilon \ell. \]

Notice that in example 2, a similar system to this one, we almost have the same conditions satisfied but \( \overline{A} \not\subseteq \overline{B} \) in that case.

**Example 4** Consider the following flexible system of linear equations, where \( \epsilon \) is a positive infinitesimal:

\[
\begin{cases}
3x + (-1 + \epsilon \O) y = 1 + \O \\
2x + y = \epsilon \ell
\end{cases}
\]

whose matricial representation is given by \( \mathcal{A} \mathcal{X} = \mathcal{B} \),

with \( \mathcal{X} = \begin{bmatrix} x \\ y \end{bmatrix} \) and \( \mathcal{A}|\mathcal{B} = \begin{bmatrix} 3 & -1 + \epsilon \O & 1 + \O \\ 2 & 1 & \epsilon \ell \end{bmatrix} \).
We have $A = \epsilon \emptyset \subset \epsilon L = B$ and

$$\Delta = \det A = \begin{vmatrix} 3 & -1 + \epsilon \emptyset \\ 2 & 1 \end{vmatrix} = 5 + \epsilon \emptyset \neq 0.$$

$\Delta$ is not an absorver of $B$ ($\Delta B = \epsilon L = B$) but $B = \epsilon L \neq \emptyset = \overline{B}$.

In this case the solution produced by Cramer’s Rule is:

$$x = \frac{\begin{vmatrix} 1 + \emptyset & -1 + \epsilon \emptyset \\ \epsilon L & 1 \end{vmatrix}}{\Delta} = \frac{1 + \emptyset}{5 + \epsilon \emptyset} = \frac{1}{5} + \emptyset$$

and

$$y = \frac{\begin{vmatrix} 3 & 1 + \emptyset \\ 2 & \epsilon L \end{vmatrix}}{\Delta} = \frac{-2 + \emptyset}{5 + \epsilon \emptyset} = -\frac{2}{5} + \emptyset.$$

We can easily see that this is not a valid solution because $\epsilon L \subset \emptyset$:

$$2x + y = \frac{2}{5} + \emptyset + \left(-\frac{2}{5} + \emptyset\right) = \emptyset \not\subseteq \epsilon L.$$
Nevertheless, if we ignore the uncertainties of the second member of system, the solution produced by Cramer’s Rule is admissible:

\[
x = \frac{1}{\Delta} \begin{vmatrix} 1 & -1 + \epsilon \otimes \\ 0 & 1 \end{vmatrix} = \frac{1}{5 + \epsilon \otimes} = \frac{1}{5} + \epsilon \otimes \quad \text{and}
\]

\[
y = \frac{1}{\Delta} \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} = -\frac{2}{5 + \epsilon \otimes} = -\frac{2}{5} + \epsilon \otimes.
\]

Now, when testing the validity of this last solution,

\[
3x + (-1 + \epsilon \otimes) y = \frac{3}{5} + \epsilon \otimes + \frac{2}{5} + \epsilon \otimes = 1 + \epsilon \otimes \subseteq 1 + \otimes \quad \text{and}
\]

\[
2x + y = \frac{2}{5} + \epsilon \otimes - \frac{2}{5} + \epsilon \otimes = \epsilon \otimes \subseteq \epsilon \mathcal{L}.
\]

**Example 5** Consider the following flexible system of linear equations, where \( \epsilon \) is a positive infinitesimal:

\[
\begin{cases} 
3x + (-1 + \epsilon^2 \otimes) y = 1 + \otimes \\
2\epsilon x + \epsilon y = \epsilon \mathcal{L}
\end{cases},
\]
whose matricial representation is given by $A\mathbf{x} = \mathbf{b}$,

with $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A|\mathbf{b} = \begin{bmatrix} 3 & -1 + \epsilon^2 \varnothing \\ 2\epsilon & \epsilon \end{bmatrix}$.

We have $\overline{A} = \epsilon^2 \varnothing \subset \epsilon \mathcal{L} = \overline{\mathbf{b}}$ and

$$
\Delta = \det A = \begin{vmatrix} 3 & -1 + \epsilon^2 \varnothing \\ 2\epsilon & \epsilon \end{vmatrix} = 5\epsilon + \epsilon^3 \varnothing \neq 0.
$$

In this case $\Delta$ is an absorver of $\overline{\mathbf{b}}$, because $\Delta \mathcal{B} = \epsilon^2 \mathcal{L} \subset \epsilon \mathcal{L} = \overline{\mathcal{B}}$, and $\overline{\mathbf{b}} = \epsilon \mathcal{L} \neq \varnothing = \overline{\mathcal{B}}$.

The solution produced by Cramer’s Rule is:

$$
x = \frac{\begin{vmatrix} 1 + \varnothing & -1 + \epsilon^2 \varnothing \\ \epsilon \mathcal{L} & \epsilon \end{vmatrix}}{\Delta} = \frac{\epsilon \mathcal{L}}{5\epsilon + \epsilon^3 \varnothing} = \mathcal{L} \text{ and }
$$

$$
y = \frac{\begin{vmatrix} 3 & 1 + \varnothing \\ 2\epsilon & \epsilon \mathcal{L} \end{vmatrix}}{\Delta} = \frac{\epsilon \mathcal{L}}{5\epsilon + \epsilon^3 \varnothing} = \mathcal{L}.
$$
We can easily see that this is not a valid solution because \( 1 + \emptyset \subset \mathcal{L} \):
\[
3x + (-1 + \epsilon^2 \emptyset) y = 3\mathcal{L} + (-1 + \epsilon^2 \emptyset) \mathcal{L} = \mathcal{L} \not\subset 1 + \emptyset.
\]

Once more, if we ignore the uncertainties of the second member of system and the uncertainty of \( \Delta \), the solution produced by Cramer’s Rule is admissible:
\[
x = \frac{1}{d} \begin{vmatrix} 1 & -1 + \epsilon^2 \emptyset \\ 0 & \epsilon \end{vmatrix} = \frac{\epsilon}{5\epsilon} = \frac{1}{5} \quad \text{and}
\]
\[
y = \frac{1}{d} \begin{vmatrix} 3 & 1 \\ 2\epsilon & 0 \end{vmatrix} = -\frac{2\epsilon}{5\epsilon} = -\frac{2}{5}.
\]

Now, when testing the validity of this last solution,
\[
3x + (-1 + \epsilon^2 \emptyset) y = \frac{3}{5} - \frac{2}{5} (-1 + \epsilon^2 \emptyset) = 1 + \epsilon^2 \emptyset \subset 1 + \emptyset \quad \text{and}
\]
\[
2\epsilon x + \epsilon y = \frac{2\epsilon}{5} - \frac{2\epsilon}{5} = 0 \subset \epsilon \mathcal{L}.
\]
4 Existence of a maximal solution

Indeed, condition \( \overline{\mathbf{A}} \subseteq \mathbf{B} \) has to be refined so that it works on the general case and it as to be replaced by condition \( \overline{\mathbf{A}} \Delta \subseteq \frac{\mathbf{B}}{\alpha^{n-1}\overline{\beta}} \), where \( \overline{\alpha} \) is the maximum coefficient on matrix \( \mathbf{A} \), \( \overline{\beta} \) is the maximum coefficient on matrix \( \mathbf{B} \) and \( n \) is the standard number of equations and variables of the system \( \mathbf{A}\mathbf{x} = \mathbf{B} \).

Definition 6 Let standard \( n \in \mathbb{N} \), \( \mathbf{A} = \left[ \alpha_{ij} \right]_{n \times n} \), with \( \alpha_{ij} = a_{ij} + A_{ij} \), and \( \mathbf{B} = \left[ \beta_j \right]_{n \times 1} \), with \( \beta_j = b_j + B_j \), we define:

1. \( \mathbf{A} \) as a non singular matrix if \( \Delta = \det \mathbf{A} \) is zeroless;

2. \( \mathbf{B} \) as a non neutricial vector if \( \overline{\beta} \) is zeroless;

3. \( R(\mathbf{A}) = \frac{\overline{\mathbf{A}}}{\Delta} \) as the relative uncertainty of \( \mathbf{A} \), if \( \mathbf{A} \) is non singular;

4. \( P(\mathbf{B}, \mathbf{A}) = \frac{\mathbf{B}}{\alpha^{n-1}\overline{\beta}} \) as the relative precision of \( \mathbf{B} \) over \( \mathbf{A} \), if \( \mathbf{A} \) is non singular and \( \mathbf{B} \) is non neutricial.
Theorem 7 Let standard $n \in \mathbb{N}$, $A = [\alpha_{ij}]_{n \times n}$ a non singular matrix, with $\alpha_{ij} = a_{ij} + A_{ij}$ and $\Delta = \det A = d + D$, and $B = [\beta_j]_{n \times 1}$ a non neutrical vector, with $\beta_j = b_j + B_j$. Consider the flexible system of linear equations $AX = B$ so that $X = [\xi_j]_{n \times 1}$, with $\xi_j = x_j + X_j$, and $R(A) \subseteq P(B, A)$. Then:

1. $X = \begin{bmatrix} \frac{\mathcal{M}_1(b)}{d} \\ \vdots \\ \frac{\mathcal{M}_n(b)}{d} \end{bmatrix}$ is a solution of $AX \subseteq B$.  

2. If $\Delta$ is not an absorver of $B$, 

$$X = \begin{bmatrix} \frac{\mathcal{M}_1(b)}{\Delta} \\ \vdots \\ \frac{\mathcal{M}_n(b)}{\Delta} \end{bmatrix}$$  

is a solution of $AX \subseteq B$.  

3. If $\Delta$ is not an absorver of $B$ and $B = B$, 

$$X = \begin{bmatrix} \frac{\mathcal{M}_1}{\Delta} \\ \vdots \\ \frac{\mathcal{M}_n}{\Delta} \end{bmatrix}$$  

is a solution of $AX \subseteq B$.  


References

